Fall 2012 Math 152

Week in Review 2

courtesy: *Oksana Shatalov*

(covering Section 7.2)

7.2: VOLUME

Key Points:

Cross sections are perpendicular to the axis of rotating. $A(x)$ is crossectional area.

- $V = \int_a^b A(x)\,\mathrm{d}x$ where $A(x)$ is the area of a moving cross-section obtained by slicing through x **perpendicular** to the x-axis.
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- $V = \int_c^d A(y)\,\mathrm{d}x$ where $A(y)$ is the area of a moving cross-section obtained by slicing through y **perpendicular** to the y-axis.
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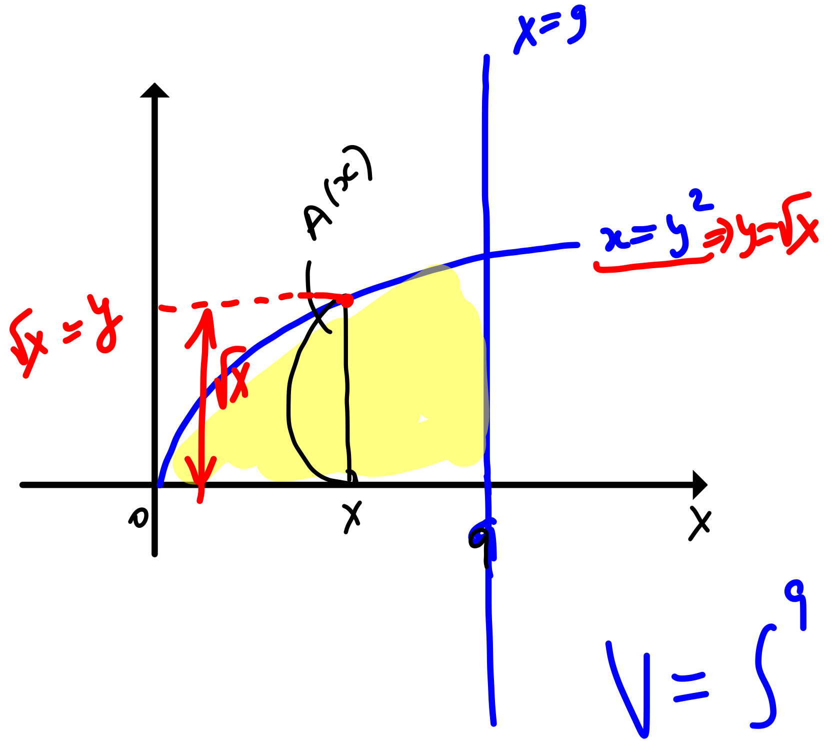
1. Find the volume of the solid whose base is the region in the first quadrant bounded by parabola $x = y^2$ and line $x = 9$ and whose cross-sections are semicircles with the diameter perpendicular to the x-axis.

$$V = \int_0^9 A(x)\,dx$$



$$A(x) = \frac{1}{2}\pi(\text{radius})^2$$

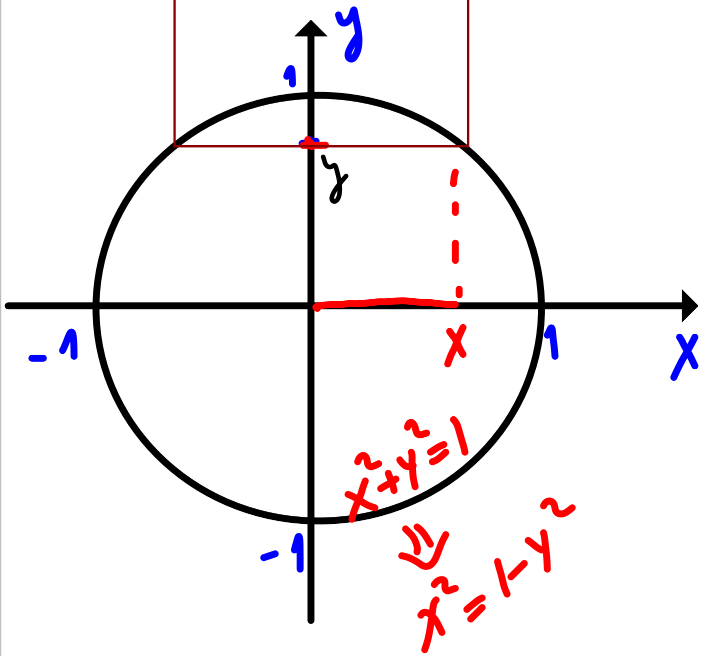
$$\frac{1}{2}\,\text{diameter} = \frac{1}{2}\sqrt{x} = \text{radius}$$

$$A(x) = \frac{1}{2}\pi\left(\frac{1}{2}\sqrt{x}\right)^2 = \frac{\pi x}{8}$$

$$V = \int_0^9 \frac{\pi x}{8} = \frac{\pi}{8}\int_0^9 x\,dx$$

$$V = \frac{\pi}{8}\,\frac{x^2}{2}\Big|_0^9 = \boxed{\frac{81\pi}{16}}$$

2. Find the volume of the solid whose base is a unit disk and whose cross-sections are squares with the base perpendicular to the y-axis.



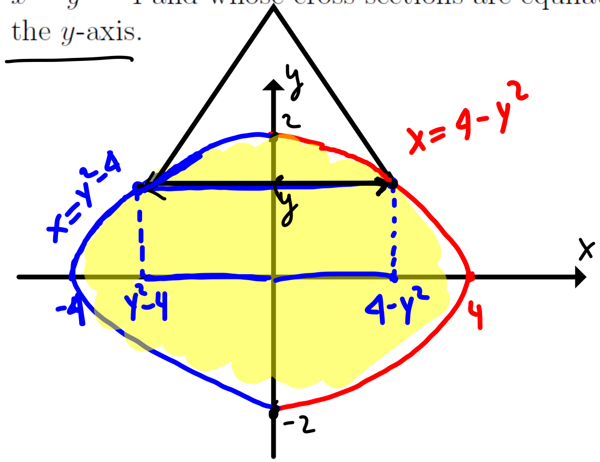
$$V=\int_{-1}^{1} A(y)\,dy$$

$$A(y)=(2x)^2=4x^2=4(1-y^2)$$

$$V=\int_{-1}^{1} 4(1-y^2)\,dy=$$

$$=4\left(y-\frac{y^3}{3}\right)\Big|_{-1}^{1}=\boxed{\frac{16}{3}}$$

3. Find the volume of the solid whose base is the region bounded by $x = 4 - y^2$ and line $x = y^2 - 4$ and whose cross-sections are equilateral triangles with the base perpendicular to the y-axis.



$$V = \int_{-2}^{2} A(y)\,dy$$

$$a = 4 - y^2 - (y^2 - 4) = 8 - 2y^2 = 2(4 - y^2)$$

Reminder

$$\boxed{A = \frac{a^2\sqrt{3}}{4}}$$

$$A(y) = \frac{a^2\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\underbrace{(8 - 2y^2)^2}_{} =$$

$$= \frac{\sqrt{3}}{4}\left(2(4 - y^2)\right)^2 = \frac{\sqrt{3}}{4}\,4(4 - y^2)^2$$

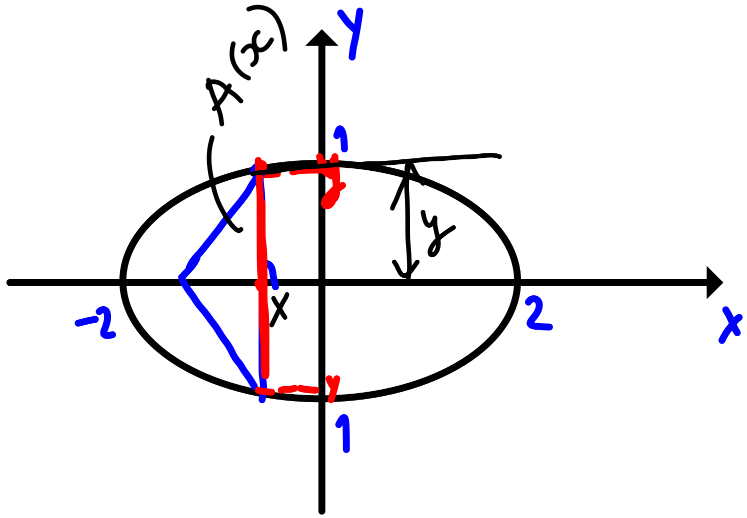
$$A(y) = \sqrt{3}\,(4 - y^2)^2$$

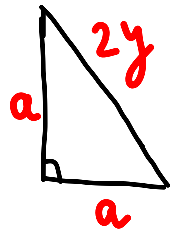
$$V = \int_{-2}^{2} \sqrt{3}(4 - y^2)^2\,dy =$$

$$= \sqrt{3}\int_{-2}^{2} 16 - 8y^2 + y^4\,dy = \boxed{\frac{512\sqrt{3}}{15}}$$

4. Determine the volume of the solid whose base is an elliptical region with boundary curve $\frac{x^2}{4} + y^2 = 1$ and whose cross-sections perpendicular to the x axis are isosceles right triangles with hypotenuse in the base.

$$V = \int_{-2}^{2} A(x)\,dx$$





$$A(x) = \frac{a^2}{2} = \frac{2y^2}{2} = y^2 = 1 - \frac{x^2}{4}$$

$$a^2 + a^2 = (2y)^2$$

$$2a^2 = 4y^2$$

$$a^2 = 2y^2$$

$$\frac{x^2}{4} + y^2 = 1$$

$$y^2 = 1 - \frac{x^2}{4}$$

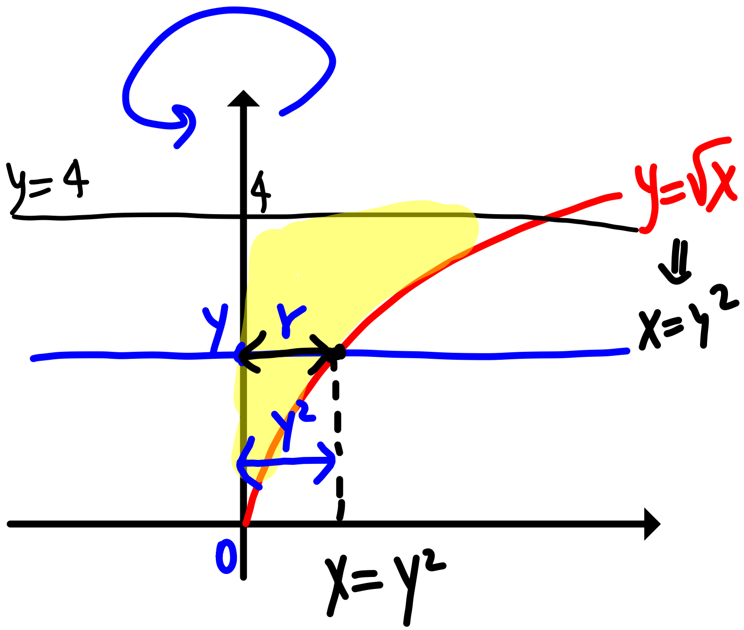
$$V = \int_{-2}^{2} \left(1 - \frac{x^2}{4}\right) dx = x - \frac{x^3}{12} \Big|_{-2}^{2} = \boxed{\frac{8}{3}}$$

Key Points: *DISK METHOD*

Cross sections perpendicular to the axis of rotating are **disks** *and* $A(x) = \pi(radius)^2$.

- A solid is formed by rotating a curve about a horizontal axis $(y = k)$: $A(x) = \pi\left[r(x)\right]^2$.
- A solid is formed by rotating a curve about a vertical axis $(x = k)$: $A(y) = \pi\left[r(y)\right]^2$.

5. Determine the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 4$ and the y-axis about the y-axis.



$$V=\int_0^4 A(y)\,dy=\pi\int_0^4 (r(y))^2\,dy$$

$$r(y)=y^2$$

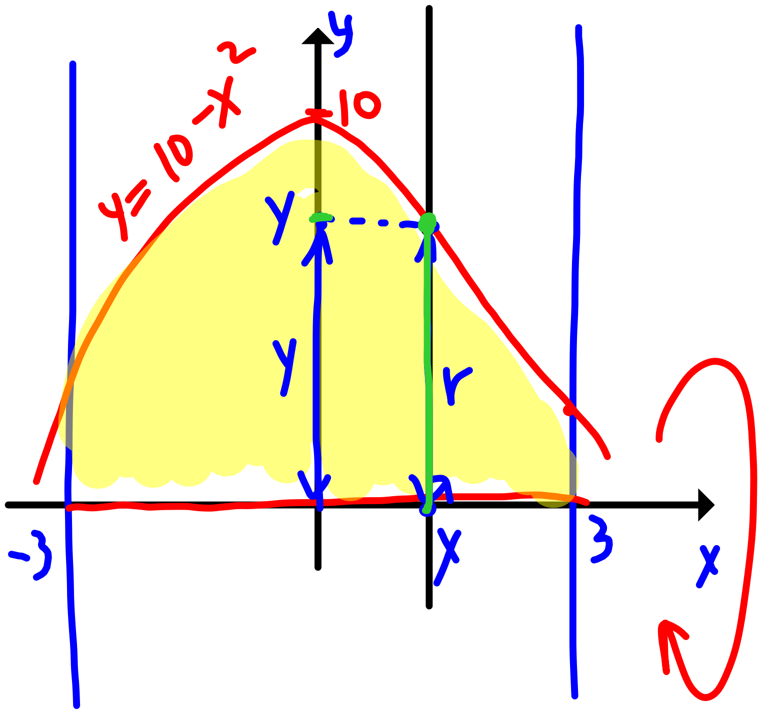
$$V=\pi\int_0^4 (y^2)^2\,dy=\pi\int_0^4 y^4\,dy$$

$$V=\frac{\pi y^5}{5}\Big|_0^4=\boxed{\frac{512\pi}{3}}$$

6. Determine the volume of the solid obtained by rotating the region

$$D = \{(x,y) | \quad -3 \le x \le 3, \quad 0 \le y \le 10 - x^2\}$$

about the x-axis.



$$V = \int_{-3}^{3} A(x)\,dx = \pi \int_{-3}^{3} [r(x)]^2\,dx$$

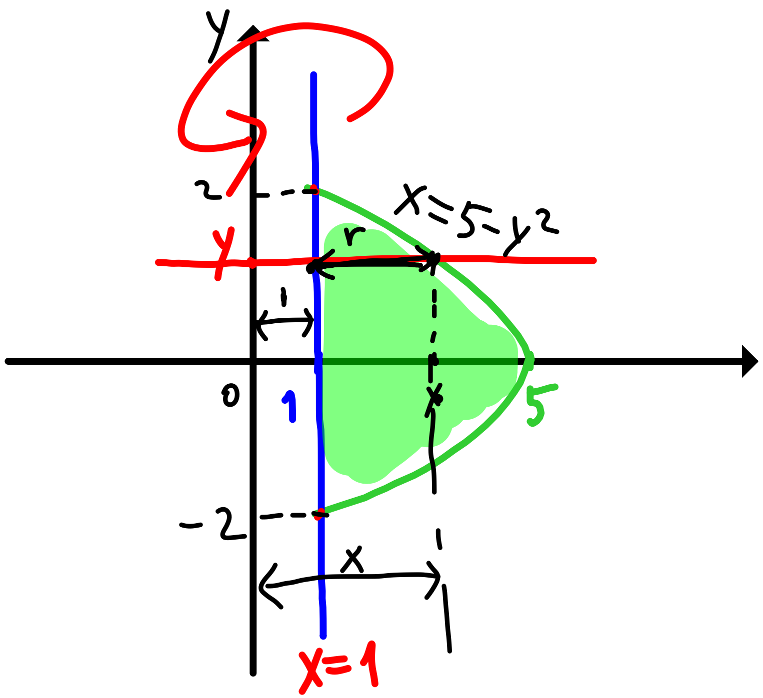
$$r(x) = y = 10 - x^2$$

$$V = \pi \int_{-3}^{3} (10 - x^2)^2\,dx$$

$$V = \pi \int_{-3}^{3} 100 - 20x^2 + x^4\,dx$$

$$V = \boxed{\frac{1686\pi}{5}}$$

7. Determine the volume of the solid obtained by rotating the region bounded by $x = 5 - y^2$, $x = 1$ about the line $x = 1$.



$$V = \int_{-2}^{2} A(y)\,dy = \pi \int_{-2}^{2} [r(y)]^2\,dy$$

$$r(y) = x - 1 = 5 - y^2 - 1 = 4 - y^2$$

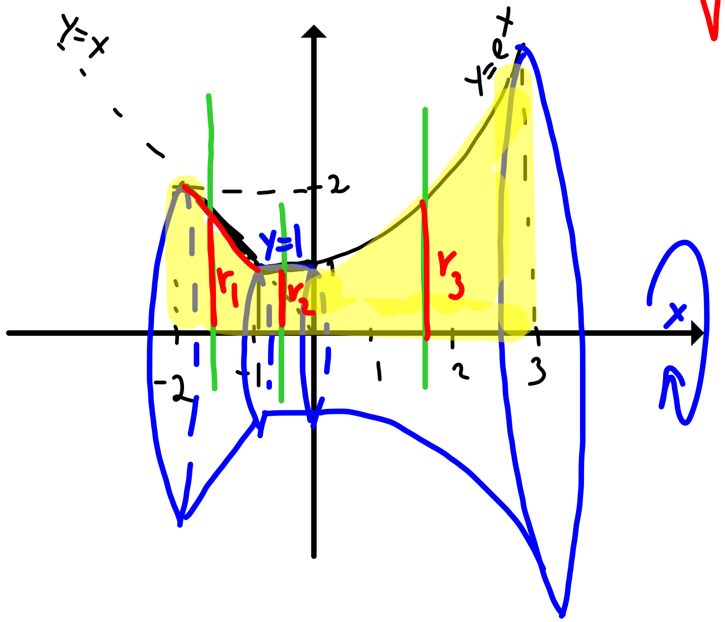
$$V = \pi \int_{-2}^{2} (4 - y^2)^2\,dy = \boxed{\frac{512\pi}{15}}$$

$$\left.\begin{array}{l} x = 5 - y^2 \\ x = 1 \end{array}\right\} \Rightarrow 1 = 5 - y^2$$

$$y^2 = 4 \Rightarrow y = \pm 2$$

8. Sketch and find the volume of the solid obtained by rotating the region under the graph of

$$f(x) = \begin{cases} -x & \text{if} \quad -2 \le x < -1 \\ 1 & \text{if} \quad -1 \le x \le 0 \\ e^x & \text{if} \quad 0 < x \le 3 \end{cases} \quad \text{about the } x\text{-axis.}$$



$$V = \int_{-2}^{-1} A_1(x)\,dx + \int_{-1}^{0} A_2(x)\,dx + \int_{0}^{3} A_3(x)\,dx$$

$$V = \pi\left[\int_{-2}^{-1} [r_1(x)]^2\,dx + \int_{-1}^{0} [r_2(x)]^2\,dx + \int_{0}^{3} [r_3(x)]^2\,dx\right]$$

$r_1(x) = y = -x$

$r_2(x) = y = 1$

$r_3(x) = y = e^x$

e^{2x}

$$V = \pi\left[\int_{-2}^{1} (-x)^2\,dx + \int_{-1}^{0} 1^2\,dx + \int_{0}^{3} (e^x)^2\,dx\right]$$

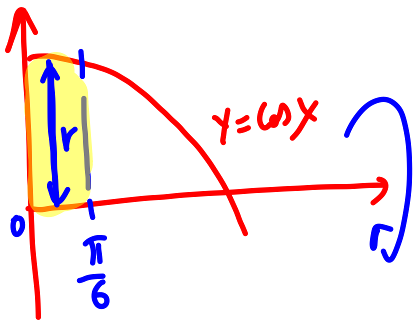
$$V = \pi\left[\frac{x^3}{3}\Big|_{-2}^{1} + 1 + \frac{e^{2x}}{2}\Big|_{0}^{3}\right] = \boxed{\pi\left[\frac{17}{6} + \frac{e^6}{2}\right]}$$

9. Each integral represents the volume of a solid. Describe the solid.

(a) $\pi \int_0^{\pi/6} \cos^2 x \, dx = V$

$$V = \pi \int_a^b (r(x))^2 \, dx$$

$y = \cos x, \quad 0 \le x \le \frac{\pi}{6}$



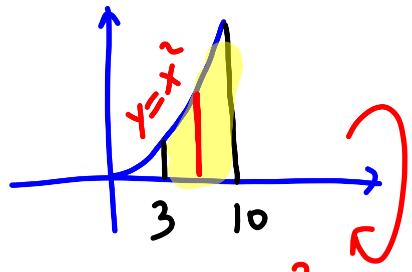
Rotate the region

$\{(x, y) \mid 0 \le x \le \frac{\pi}{6}, \ 0 \le y \le \cos x\}$

about the x-axis

(b) $\pi \int_3^{10} x^4 \, dx = \pi \int_3^{10} [r(x)]^2 \, dx$

$y = x^2$



$r(x) = y = x^2$

Rotate the region

$\{(x, y) : 3 \le x \le 10, \ 0 \le y \le x^2\}$

about the x-axis

Key Points: *WASHER METHOD*

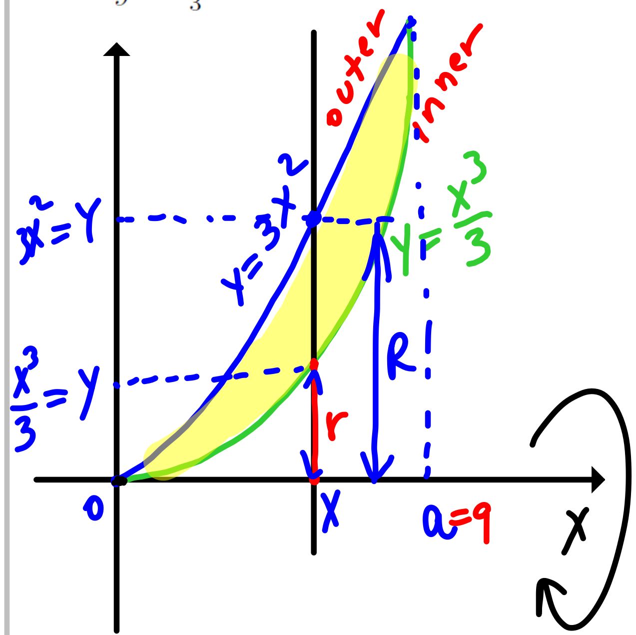
Cross sections perpendicular to the axis of rotating are **washers** *(rings) and*

$$A(x) = \pi \left[\begin{pmatrix} outer \\ radius \end{pmatrix}^2 - \begin{pmatrix} inner \\ radius \end{pmatrix}^2 \right]$$

.

- If a solid is formed by rotating about a horizontal axis $(y = k)$ then $A(x) = \pi \left[R(x)^2 - r(x)^2 \right]$
- If a solid is formed by rotating about a vertical axis $(x = k)$ then $A(y) = \pi \left[R(y)^2 - r(y)^2 \right]$

10. Determine the volume of the solid obtained by rotating the region bounded by $y = 3x^2$ and $y = \frac{x^3}{3}$ about the x-axis.



$$V = \int_0^a A(x)\,dx = \pi \int_0^a [R(x)]^2 - [r(x)]^2\,dx$$

Find a: $3x^2 = \frac{x^3}{3} \Rightarrow 9x^2 - x^3 = 0$

$x^2(9-x) = 0 \rightarrow x = 0$, $x = \boxed{9 = a}$

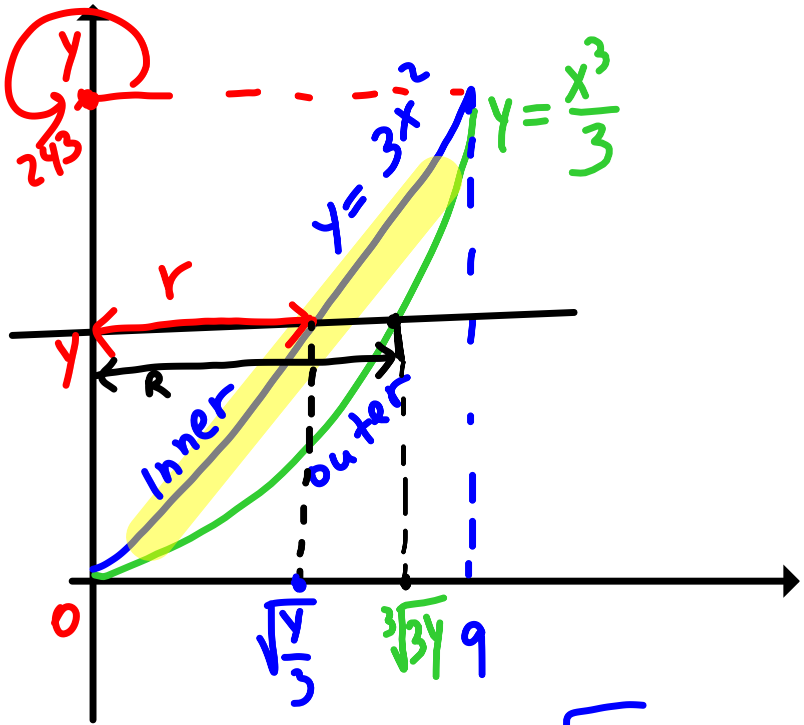
$r(x) = y = \frac{x^3}{3}$

$R(x) = y = 3x^2$

$$V = \pi \int_0^9 (3x^2)^2 - \left(\frac{x^3}{3}\right)^2 dx = \pi \int_0^9 9x^4 - \frac{x^6}{9}\,dx =$$

$$= \boxed{\frac{2\pi}{35} \cdot 9^6} = \frac{1062882\pi}{35}$$

11. Determine the volume of the solid obtained by rotating the region bounded by $y = 3x^2$ and $y = \frac{x^3}{3}$ about the y-axis.



inner: $y=3x^2 \Rightarrow x=\sqrt{\frac{y}{3}}$

outer: $y=\frac{x^3}{3} \Rightarrow x=\sqrt[3]{3y}$

$$V=\int_0^{243} A(y)\,dy$$

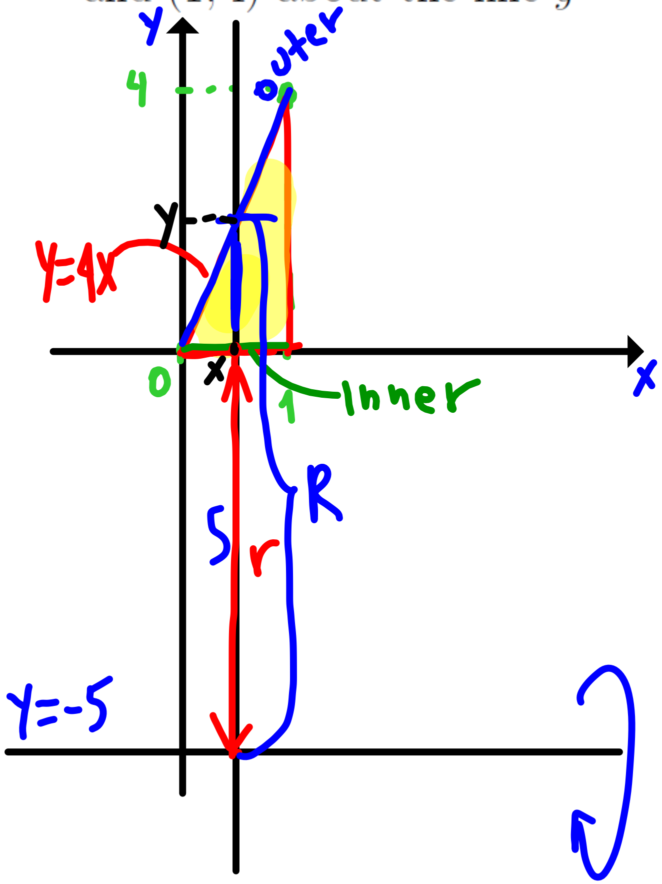
$$V=\pi\int_0^{243} [R(y)]^2-[r(y)]^2\,dy$$

$$R(y)=\sqrt[3]{3y}=(3y)^{\frac{1}{3}}$$

$$r(y)=\sqrt{\frac{y}{3}}=\left(\frac{y}{3}\right)^{\frac{1}{2}}$$

$$V=\pi\int_0^{243}(3y)^{\frac{2}{3}}-\frac{y}{3}\,dy=\frac{85293}{10}\pi$$

12. Determine the volume of the solid obtained by rotating triangle with vertices $(0,0)$, $(1,0)$ and $(1,4)$ about the line $y=-5$.



$$V=\int_0^1 A(x)\,dx=\pi\int_0^1 [R(x)]^2-[r(x)]^2\,dx$$

$r(x)=5$

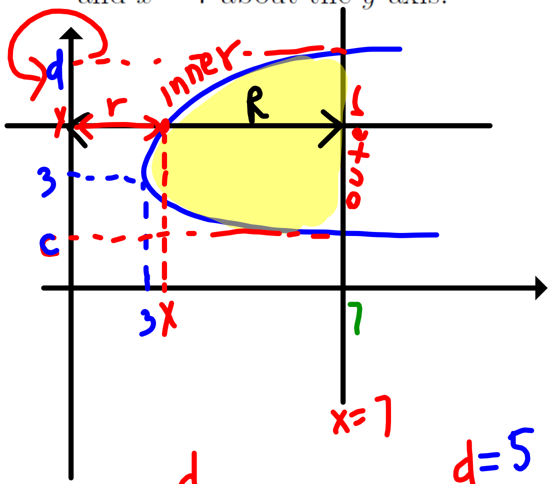
$R(x)=5+y=5+4x$

$$V=\pi\int_0^1 (5+4x)^2-5^2\,dx$$

$$V=\pi\int_0^1 \cancel{25}+16x^2+40x-\cancel{25}\,dx$$

$$V=\pi\left(\frac{16x^3}{3}+20x^2\right)\Big|_0^1=\pi\left(\frac{16}{3}+20\right)=\boxed{\frac{76\pi}{3}}$$

13. Determine the volume of the solid obtained by rotating the region bounded by $x = y^2 - 6y + 12$ and $x = 7$ about the y-axis.



Find c & d

$y^2 - 6y + 12 = 7$

$y^2 - 6y + 5 = 0$

$(y-1)(y-5) = 0$

$y = 1$ or $y = 5$

$\quad\;\; \| \qquad\qquad \|$

$\quad\;\; c \qquad\qquad d$

$x = y^2 - 6y + 9 + 3$

$x = (y-3)^2 + 3$

$R(y) = 7$

$r(y) = x = y^2 - 6y + 12$

$$V = \int_c^d A(y)\,dy = \pi \int_{c=1}^{d=5} [R(y)]^2 - [r(y)]^2\,dy =$$

$$= \pi \int_1^5 7^2 - (y^2 - 6y + 12)^2\,dy =$$

$$= \pi \int_1^5 49 - (y^4 + 36y^2 + 144 - 12y^3 + 24y^2 - 144y)\,dy$$

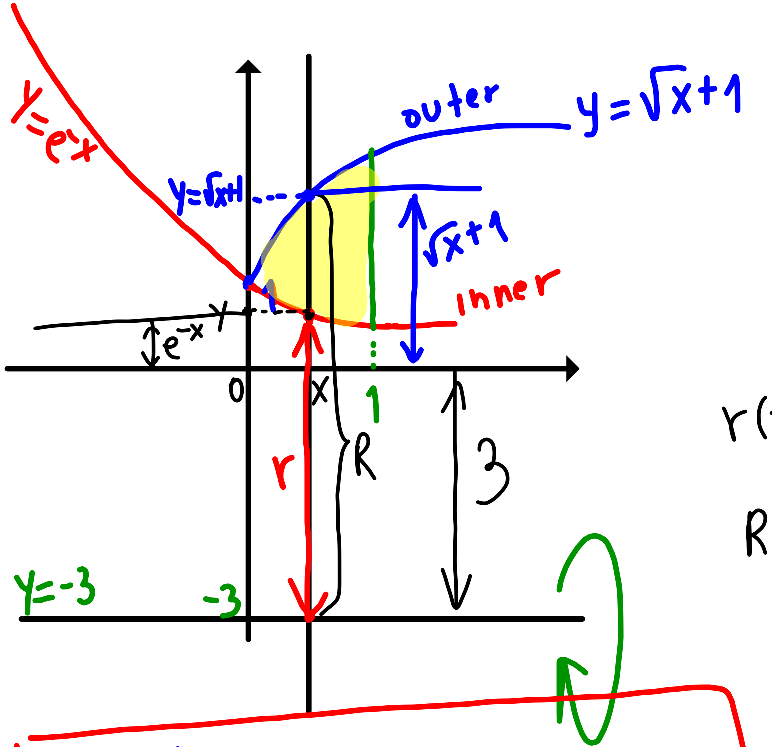
$$= \ldots = \frac{477\pi}{5}$$

You can use FOIL

or

$(a \pm b \pm c)^2 = a^2 + b^2 + c^2 \pm 2ab \pm 2ac \pm 2bc$

14. Set up the integral representing the volume of the solid obtained by rotating the region bounded by $y = e^{-x}$, $y = \sqrt{x} + 1$ and $x = 1$ about the line $y = -3$.



$$V = \int_0^1 A(x)\,dx$$

$$V = \pi \int_0^1 [R(x)]^2 - [r(x)]^2\,dx$$

$$r(x) = 3 + e^{-x}$$

$$R(x) = \sqrt{x} + 1 + 3 = 4 + \sqrt{x}$$

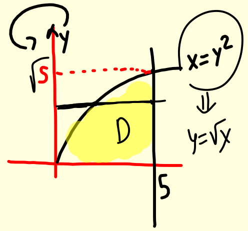
$$\boxed{V = \pi \int_0^1 (4+\sqrt{x})^2 - (3+e^{-x})^2\,dx}$$

$$V = \pi \int_0^1 16 + 8\sqrt{x} + x - (9 + 6e^{-x} + e^{-2x})\,dx$$

15. Each integral represents the volume of a solid. Describe the solid.

(a) $\pi \int_0^{\sqrt{5}} (\underbrace{25}_{=5^2} - \underbrace{y^4}_{=(y^2)^2})\, \underbrace{dy}_{\text{Rotation about y-axis}} = \pi \int_0^{\sqrt{5}} [R(y)]^2 - [r(y)]^2 dy$

$R(y) = \underset{\overset{\|}{x}}{5}$, $r(y) = \underset{\overset{\|}{x}}{y^2}$

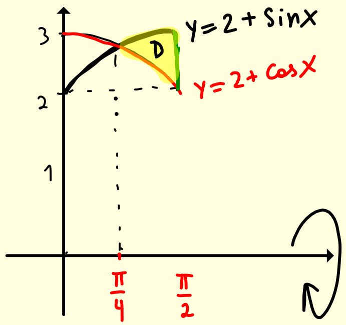


Rotate the region

$D = \{ (x,y) \mid 0 \le x \le 5,\ 0 \le y \le \sqrt{x} \}$

about the y-axis

(b) $\pi \int_{\pi/4}^{\pi/2} [(\underbrace{2+\sin x}_{R(x)})^2 - (\underbrace{2+\cos x}_{r(x)})^2]\, \underbrace{dx}_{x\text{-axis}}$



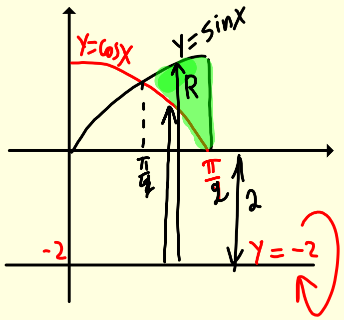
Rotate the region

$D = \{ (x,y) : \frac{\pi}{4} \le x \le \frac{\pi}{2},$

$2+\cos x \le y \le 2+\sin x \}$

about the x-axis.

Another description



Rotate the region

$R = \{ (x,y) \mid \frac{\pi}{4} \le x \le \frac{\pi}{2},\ \cos x \le y \le \sin x \}$

about the line $y=-2$.