Fall 2012 Math 152

Week in Review 4

courtesy: *Oksana Shatalov*

(covering Sections 8.1&8.2 and Exam 1 Review)

8.1: Integration By Parts

Key Points

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

$$\int_a^b u\,\mathrm{d}v = uv\big|_a^b - \int_a^b v\,\mathrm{d}u \quad \text{where} \quad uv\big|_a^b = u(b)v(b) - u(a)v(a).$$

To choose u use the LIATE rule, in order of preference for u:

Logarithms

Inverse Trigonometric Functions

Algebraic, powers of x

Trigonometric functions

Exponential functions

1. Compute the following integrals

(a) $I = \displaystyle\int \frac{x}{3} \sin \frac{x-1}{3} \, \mathrm{d}x$

$u = \frac{x}{3}$ $\qquad$ $\sin \frac{x-1}{3} dx = dV$

$du = \frac{dx}{3}$ $\qquad$ $V = \int \sin \frac{x-1}{3} dx = -3 \cos \frac{x-1}{3}$

$$I = uv - \int v\,du = -\frac{x}{\cancel{3}} \cdot \cancel{3} \cos \frac{x-1}{3} - \int -\cancel{3} \cos \frac{x-1}{3} \frac{dx}{\cancel{3}}$$

$$= -x \cos \frac{x-1}{3} + \int \cos \frac{x-1}{3} dx$$

$$= -x \cos \frac{x-1}{3} + 3 \sin \frac{x-1}{3} + C$$

(b) $I = \int_0^1 \frac{x^2}{e^{10x}}\,dx = \int_0^1 x^2 e^{-10x}\,dx$

1st

$u = x^2 \qquad dv = e^{-10x}\,dx$

$du = 2x\,dx \qquad v = -\frac{1}{10}e^{-10x}$

$$I = uv - \int v\,du = -\frac{1}{10}x^2 e^{-10x} - \int -\frac{1}{10}e^{-10x}\cdot 2x\,dx$$

$$I = -\frac{x^2}{10}e^{-10x} + \frac{1}{5}\underbrace{\int x e^{-10x}\,dx}$$

2nd

$u = x \qquad e^{-10x}\,dx = dv$

$du = dx \qquad v = -\frac{1}{10}e^{-10x}$

$$I = -\frac{x^2}{10}e^{-10x} + \frac{1}{5}\left[-\frac{x}{10}e^{-10x} - \int -\frac{1}{10}e^{-10x}\,dx\right]$$

$$I = -\frac{x^2}{10}e^{-10x} - \frac{x}{50}e^{-10x} + \frac{1}{50}\int e^{-10x}\,dx$$

$$I = -\frac{x^2}{10}e^{-10x} - \frac{x}{50}e^{-10x} + \frac{1}{50}\cdot\left(-\frac{1}{10}\right)e^{-10x}$$

$$I = -\frac{e^{-10x}}{10}\left[+x^2 + \frac{x}{5} + \frac{1}{50}\right]\Bigg|_0^1 =$$

Plug in bounds $\rightarrow$

$$= -\left[\frac{e^{-10}}{10}\left[1 + \frac{1}{5} + \frac{1}{50}\right] - \frac{1}{10}\left[0 + 0 + \frac{1}{50}\right]\right]$$

$$= \frac{1}{500}\left(1 - \frac{61}{e^{10}}\right)$$

(c) $I = \int \arctan \frac{1}{x} \, dx$

$u = \arctan \frac{1}{x}$ $\quad dv = dx \implies v = x$

$du = \frac{1}{1+\frac{1}{x^2}} \cdot \left(\frac{1}{x}\right)' \cancel{dx} = -\frac{\frac{1}{\cancel{x^2}} dx}{\frac{x^2+1}{\cancel{x^2}}} = -\frac{dx}{x^2+1}$

$I = uv - \int v\,du = x \arctan \frac{1}{x} + \int \frac{x\,dx}{x^2+1}$

u-sub.

$u = x^2+1$

$du = 2\underbrace{x\,dx}$

$I = x\arctan\frac{1}{x} + \int \frac{du}{2u} = x\arctan\frac{1}{x} + \frac{1}{2}\ln|u|$

$I = x\arctan\frac{1}{x} + \frac{1}{2}\ln(x^2+1) + C$

8.2: Trigonometric Integrals

Key Points

- How to evaluate $\int \sin^n x \cos^m x \, dx$

sin2x+cos2x=1

1. *If n is odd use substitution $u = \cos x$ (Strip out one sine and convert the rest to cosine.)*
2. *If m is odd use substitution $u = \sin x$ (Strip out one cosine and convert the rest to sine.)*
3. *If both n and m are odd use 1 or 2.*
4. *If both n and m are even, use the half-angle identities:*

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x); \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x); \qquad \sin x \cos x = \frac{1}{2}\sin 2x$$

- [1] How to evaluate $\int \sec^n x \tan^m x \, dx$

1. *If n is even use formula $\sec^2 x = 1 + \tan^2 x$ and substitution $u = \tan x \quad \Rightarrow \quad du = \sec^2 x dx$.*
2. *If m is odd use formula $\tan^2 x = \sec^2 x - 1$ and substitution $u = \sec x \quad \Rightarrow \quad du = \sec x \tan x dx$*

[1]Integral $\int \csc^n x \cot^m x \, dx$ can be found by similar methods because of the identity $1 + \cot^2 x = \csc^2 x$.

- How to evaluate $\int \sin(Ax)\cos(Bx) \, dx$, $\int \sin(Ax)\sin(Bx) \, dx$, $\int \cos(Ax)\cos(Bx) \, dx$

Use the following identities:(The identities below need not be memorized for the exam)

$$\begin{aligned} \sin A \cos B &= \tfrac{1}{2}(\sin(A-B) + \sin(A+B)) \\ \sin A \sin B &= \tfrac{1}{2}(\cos(A-B) - \cos(A+B)) \\ \cos A \cos B &= \tfrac{1}{2}(\cos(A-B) + \cos(A+B)) \end{aligned}$$

Examples

2. Compute the following integrals

(a) $I = \int \frac{\sec^4 x}{\tan^7 x}\,dx = \int \frac{\sec^2 x \overbrace{\sec^2 x\,dx}^{d(\tan x)}}{\tan^7 x}$

$$\sec^2 x = 1 + \tan^2 x$$

$$I = \int \frac{1+\tan^2 x}{\tan^7 x} \sec^2 x\,dx$$

$$u = \tan x \Rightarrow du = \sec^2 x\,dx$$

$$I = \int \frac{1+u^2}{u^7}\,du = \int u^{-7} + u^{-5}\,du =$$

$$= \frac{u^{-6}}{-6} + \frac{u^{-4}}{-4} + C = -\frac{1}{6u^6} - \frac{1}{4u^4} + C$$

$$I = -\frac{1}{6\tan^6 x} - \frac{1}{4\tan^4 x} + C$$

(b) $I = \int \sin^5 x \sqrt{\cos x}\, dx = \int \underbrace{\sin^4 x}_{\substack{(\sin^2 x)^2 \\ \parallel \\ (1-\cos^2 x)^2}} \sqrt{\cos x}\ \underbrace{\sin x\, dx}_{-d(\cos x)}$

$u = \cos x\ ,\ du = -\sin x\, dx$

$(a-b)^2 = a^2 - 2ab + b^2$

$I = -\int (1-u^2)^2 \sqrt{u}\, du = -\int (1 - 2u^2 + u^4)\, u^{\frac{1}{2}}\, du$

$I = -\int u^{\frac{1}{2}} - 2u^{5/2} + u^{9/2}\, du$

$I = -\frac{2}{3} u^{3/2} + 2 \cdot \frac{2}{7} u^{7/2} - \frac{2}{11} u^{11/2} + C$

$u = \cos x$

$I = -\frac{2}{3} \cos^{3/2} x + \frac{4}{7} \cos^{7/2} x - \frac{2}{11} \cos^{11/2} x + C$

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(c) $I = \int \frac{\cos^5(\ln x)}{x} \mathrm{d}x$

$u = \ln x \Rightarrow du = \frac{dx}{x}$

$$I = \int \cos^5 u \, du = \int \underbrace{\cos^4 u}_{\substack{(\cos^2 u)^2 \\ (1-\sin^2 u)^2}} \underbrace{\cos u \, du}_{d(\sin u)}$$

$U = \sin u \Rightarrow dU = \cos u \, du$

$$I = \int (1-U^2)^2 \, dU = \int (1 - 2U^2 + U^4) \, dU$$

$$I = U - \frac{2}{3} U^3 + \frac{U^5}{5} + C$$

$$I = \sin u - \frac{2}{3} \sin^3 u + \frac{1}{5} \sin^5 u + C$$

$$I = \sin(\ln x) - \frac{2}{3} \sin^3(\ln x) + \frac{1}{5} \sin^5(\ln x) + C$$

(d) $\int \cos^2(3x)\sin^2(3x)\,dx = \int \underbrace{\left(\cos(3x)\sin(3x)\right)}^2 dx$

$\cos A \sin A = \frac{\sin(2A)}{2}$

$A = 3x$ $\int \left(\frac{\sin(2\cdot 3x)}{2}\right)^2 dx$

$\frac{1}{4}\int \sin^2(6x)\,dx$

$\sin^2 B = \frac{1}{2}(1-\cos(2B))$

$B = 6x$

$\frac{1}{4}\int \frac{1}{2}(1-\cos(12x))\,dx$

$\frac{1}{8}\int 1-\cos(12x)\,dx$

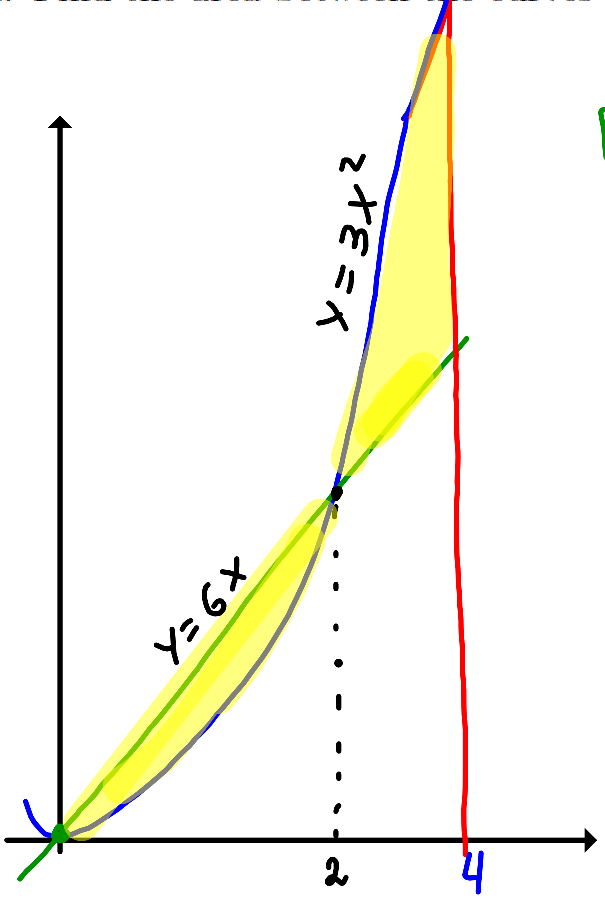
$\frac{1}{8}\left(x - \frac{1}{12}\sin(12x)\right) + C$

General Exam 1 Review

3. If $F(x) = \int_0^{e^x} \cos(t^2)\,dt$ what is $F'(x)$?

$$F'(x) = \cos\left((e^x)^2\right) \cdot (e^x)' = e^x \cos\left(e^{2x}\right)$$

4. Find the area between the curves $y = 3x^2$ and $y = 6x$ from $x = 0$ to $x = 4$.



Find intersection points

$3x^2 = 6x$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$ or $x = 2$

$$\text{Area} = \int_a^b (\text{Top}) - (\text{Bottom})\, dx$$

$$\text{Area} = \int_0^2 (6x - 3x^2)\, dx$$

$$+ \int_2^4 (3x^2 - 6x)\, dx$$

$$\text{Area} = 3x^2 - x^3 \Big|_0^2 + (x^3 - 3x^2)\Big|_2^4$$

$$\text{Area} = 12 - 8 + 64 - 48 - 8 + 12 = \boxed{24}$$

5. Compute $\int_0^1 \frac{2x}{\sqrt[4]{x^2+1}}\,dx$. $= \int_1^2 \frac{du}{\sqrt[4]{u}} = \int_1^2 u^{-\frac{1}{4}}\,du$

$u = x^2+1 \Rightarrow du = 2x\,dx$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=2$

$$= \left.\frac{u^{-\frac{1}{4}+1}}{-\frac{1}{4}+1}\right|_1^2 = \left.\frac{4}{3}u^{3/4}\right|_1^2 =$$

$$\boxed{\frac{4}{3}\left(2^{3/4}-1\right)}$$

6. After an appropriate substitution, the integral $\int_{-1}^{2} 3x^5\sqrt{x^3+1}\,dx$ is equivalent to which of the following?

(a) $\int_0^3 \left(u^{3/2} - \cancel{u^{1/2}}\right) du$ (b) $\int_{-1}^2 \left(\cancel{u^{1/2} - u^{3/2}}\right) du$ (c) $\int_0^3 \left(\cancel{u^{1/2}} + u^{3/2}\right) du$

(d) $\int_0^9 \left(u^{3/2} - u^{1/2}\right) du$ (e) $\int_{-1}^2 \cancel{x^3 u^{1/2}}\, du$

$u = \underbrace{x^3+1} \Rightarrow du = \underbrace{3x^2 dx}$

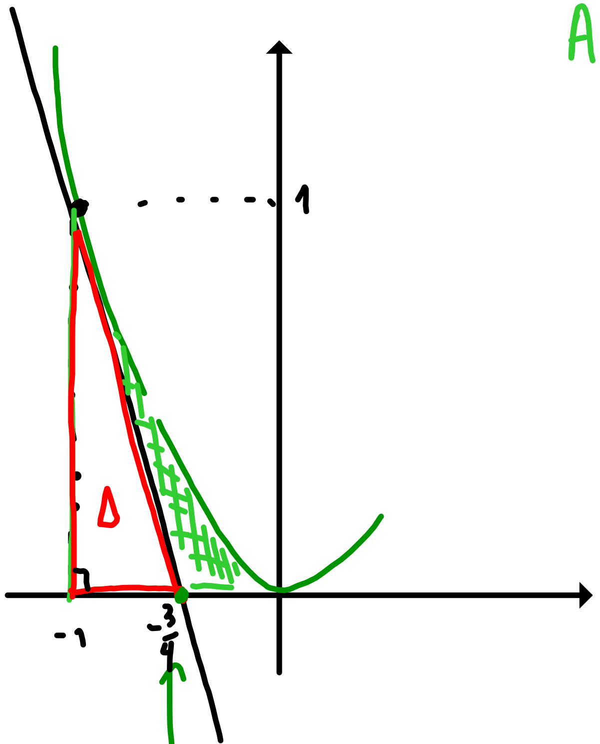
$$\int_{-1}^{2} \underbrace{x^3}_{u-1} \cdot \underbrace{\sqrt{x^3+1}}_{\sqrt{u}}\ \underbrace{3x^2 dx}_{du} = \int_0^9 (u-1)\,u^{\frac{1}{2}}\,du = \int_0^9 u^{3/2} - u^{1/2}\,du$$

$x = -1 \Rightarrow u = 0$

$x = 2 \Rightarrow u = 9$

7. Determine the area of the region bounded by the x-axis, the curve $y = x^4$ and tangent line to this curve at the point $(-1, 1)$.

$$\text{Area} = \int_{-1}^{0} x^4\,dx - \text{Area}(\Delta)$$



Find tangent

$$m = (x^4)'\Big|_{x=-1} = 4x^3\Big|_{x=-1} = -4$$

$$(x_1, y_1) = (-1, 1)$$

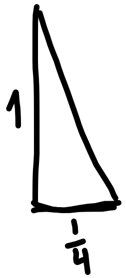
$$y - y_1 = m(x - x_1)$$

$$y - 1 = -4(x + 1)$$

$$y = -4x - 4 + 1 \Rightarrow \boxed{y = -4x - 3}$$

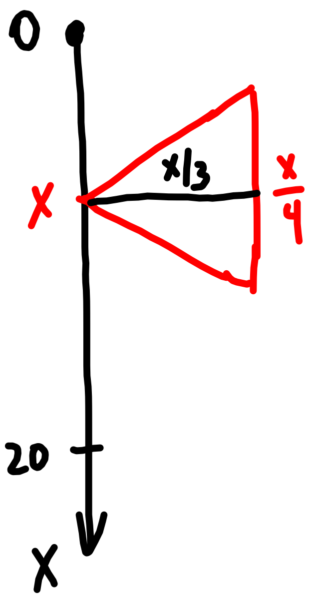
Tangent

If $y = 0 \Rightarrow x = -3/4$



$$\text{Area} = \int_{-1}^{0} x^4\,dx - \frac{1}{2}\cdot 1 \cdot \frac{1}{4} = \frac{1}{5} - \frac{1}{8} = \boxed{\frac{3}{40}}$$

8. The height of a monument is 20 m. A horizontal cross-section at a distance x meters from the top is an isosceles triangle with base $x/4$ meters and hight $x/3$ meters. Set up, but *do not evaluate*, an integral for the volume of the monument.

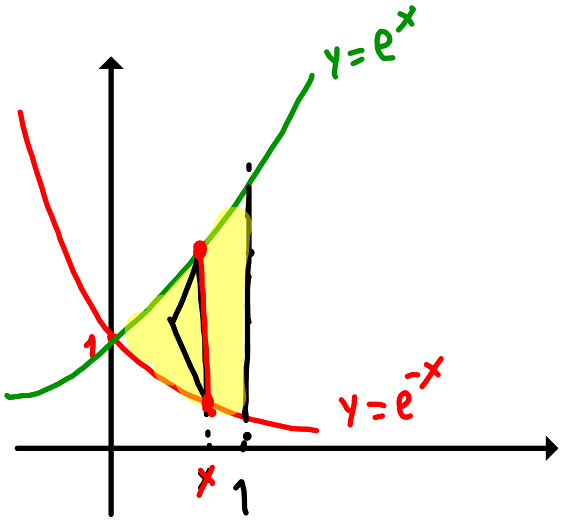


$$V = \int_0^{20} A(x)\,dx$$

$$A(x) = \frac{1}{2}\,\frac{x}{4}\cdot\frac{x}{3} = \frac{x^2}{24}$$

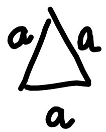
$$\boxed{V = \int_0^{20} \frac{x^2}{24}\,dx}$$

9. Find the volume of the solid whose base is the area enclosed by $y = e^x$ and $y = e^{-x}$ from $[0, 1]$ with cross-sections perpendicular to the x-axis that are equilateral triangles.



$$V = \int_0^1 A(x)\,dx$$

$$A(x) = \frac{a^2\sqrt{3}}{4}$$



$$a = e^x - e^{-x}$$

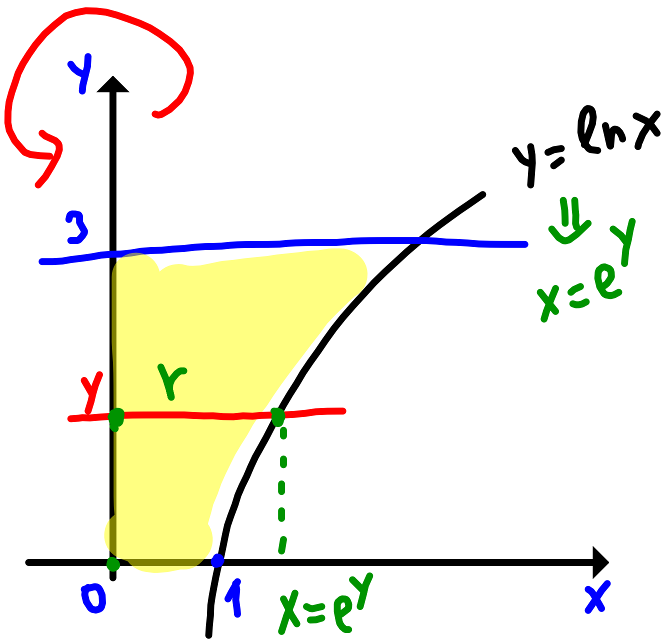
$$A(x) = \frac{\sqrt{3}}{4}\left(e^x - e^{-x}\right)^2$$

$$V = \frac{\sqrt{3}}{4}\int_0^1 \left(e^x - e^{-x}\right)^2 dx = \frac{\sqrt{3}}{4}\int_0^1 \left(e^{2x} - 2\underbrace{e^x \cdot e^{-x}}_{=1} + e^{-2x}\right) dx$$

$$V = \frac{\sqrt{3}}{4}\int_0^1 \left(e^{2x} - 2 + e^{-2x}\right) dx = \frac{\sqrt{3}}{4}\left(\frac{e^{2x}}{2} - 2x - \frac{e^{-2x}}{2}\right)\Bigg|_0^1$$

$$V = \frac{\sqrt{3}}{4}\left(\frac{e^2}{2} - 2 - \frac{e^{-2}}{2}\right)$$

10. Find the volume of the solid formed by rotating the region bounded by $x = 0$, $y = \ln(x)$, $y = 0$, and $y = 3$ about the y-axis.



Use disks method

$$V = \int_0^3 A(y)\,dy$$

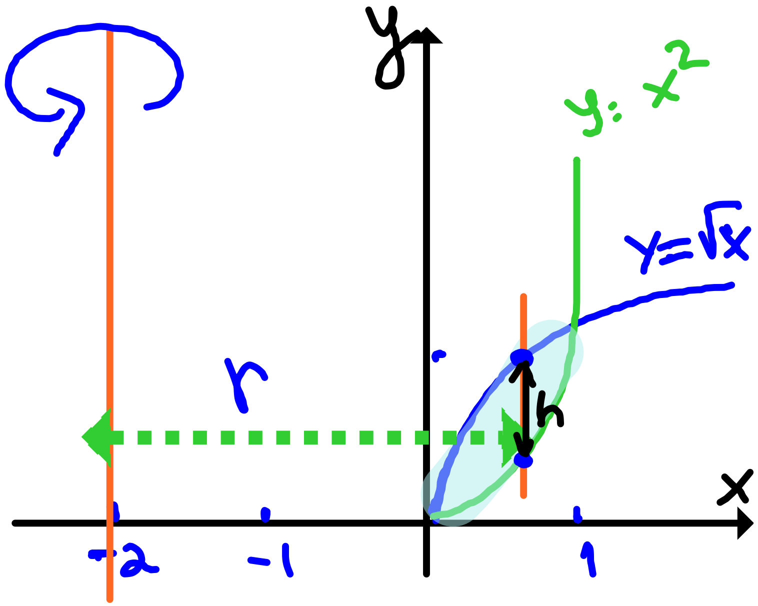
$$A(y) = \pi r^2(y) = \pi\left(e^y\right)^2 = \pi e^{2y}$$

$$V = \pi \int_0^3 e^{2y}\,dy = \frac{\pi}{2} e^{2y} \Big|_0^3 = \frac{\pi}{2}\left(e^6 - 1\right)$$

11. Using **cylindrical shells** which of the following integrals gives the volume of the solid formed by rotating the region bounded by $y = \sqrt{x}$ and $y = x^2$ about the line $x = -2$?

(a) $2\pi \int_0^1 (x-2)(\sqrt{x} - x^2)\, dx$ (b) $2\pi \int_0^1 (x+2)(\sqrt{x} - x^2)\, dx$ (c) $\pi \int_0^1 (x^2 - \sqrt{x})^2\, dx$

(d) $\pi \int_0^1 \left[(y^2-2)^2 - (\sqrt{y}-2)^2\right] dy$ (e) $\pi \int_0^1 (y^2 - \sqrt{y})\,(y+2)\, dy$

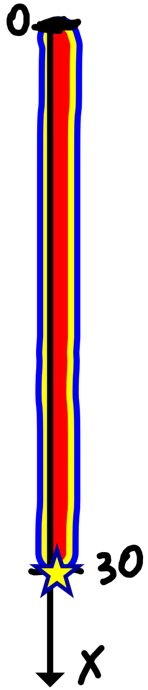


$V = \int_0^1 A(x)\,dx = 2\pi \int_0^1 r(x)\,h(x)\,dx$

$r(x) = x - (-2) = x + 2$

$h(x) = \sqrt{x} - x^2$

12. A 15-Newton weight is suspended vertically at the end of a 30 m long rope. The rope weighs 6 Newtons. How much work (in Newton-m) is required to pull the weight to the top?



$$W = W_{rope} + W_{\star}$$

$$W_{rope} = \int_0^{30} \underline{\underline{w}}\, x \, dx \qquad W_{\star} = \underbrace{(\text{weight})}_{=15N} \cdot \underbrace{(\text{dist.})}_{=30\,m}$$

$$w = \frac{6\,N}{30\,m} = \frac{1}{5}\,N/m$$

$$W = \int_0^{30} \frac{1}{5} x \, dx + 15 \cdot 30$$

$$W = \frac{1}{5} \frac{x^2}{2} \Big|_0^{30} + 450 = 540\, N\cdot m = 540\, J$$

13. If a $25J$ work is required to keep a spring $1m$ beyond its natural length, how much work is done in stretching the spring from $2m$ to $4m$ beyond its natural length?

$$W = \int_a^b kx\,dx$$

$$25 = \int_0^1 kx\,dx$$

$$\Downarrow$$

$$25 = k\frac{x^2}{2}\Big|_0^1$$

$$25 = \frac{k}{2}$$

$$\boxed{k = 50}$$

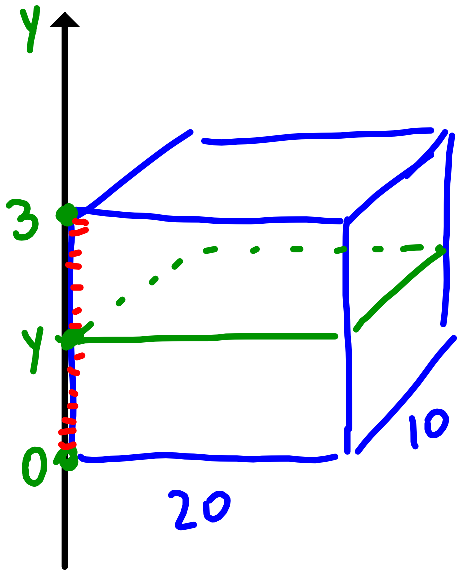
Find $W = \int_2^4 kx\,dx$

$$W = 50\int_2^4 x\,dx$$

$$W = 50\,\frac{x^2}{2}\Big|_2^4 = 25(16-4)$$

$$\boxed{W = 300J}$$

14. A rectangular swimming pool 20 m long, 10 m wide and 3 m deep is full of water (density = ρ kg/m^3). What is the work required to pump all the water out of the top of the pool? (Leave your answer in terms of density ρ and the gravitational constant g.)



$$W = \rho g \int_0^3 A(y)\,\text{dist}(y)\,dy$$

$$A(y) = 20 \cdot 10 = 200$$

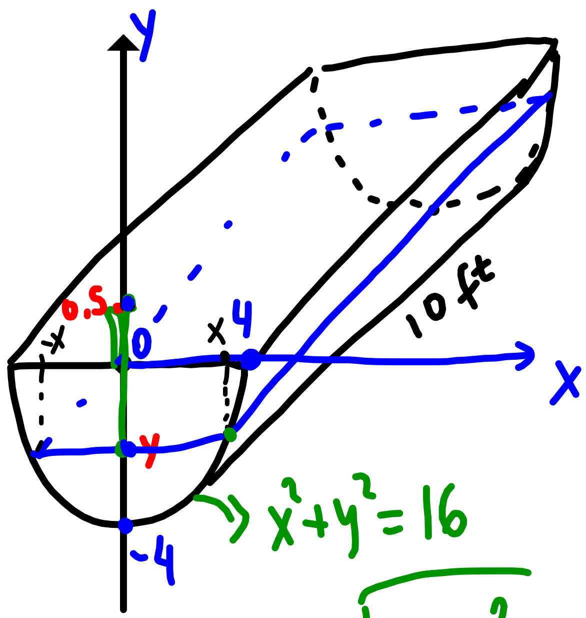
$$\text{dist}(y) = 3 - y$$

$$W = \rho g \int_0^3 200(3-y)\,dy$$

$$W = 200\rho g \left(3y - \frac{y^2}{2}\right)\Big|_0^3$$

$$W = 200\rho g \left(9 - \frac{9}{2}\right) = 900\rho g$$

15. A tank of water is ~~is~~ a trough 10 feet long and has a vertical cross section in the shape of a semi circle with radius 4 feet, diameter at the top. The tank is filled with water. The water is pumped from a spout at the top of the tank that is 0.5 ft high. Find the work needed to pump out the water until the water level is 1m from the bottom. Just set up the integral (assume that the density of water is 62.5lb/ft^3.)



$$W = \rho g \int_{-3}^{0} A(y)\, dist(y)\, dy$$

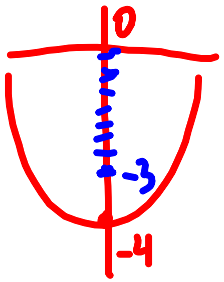
$$A(y) = 10 \cdot 2x$$

$$x^2 + y^2 = 16$$

$$x = \sqrt{16 - y^2}$$

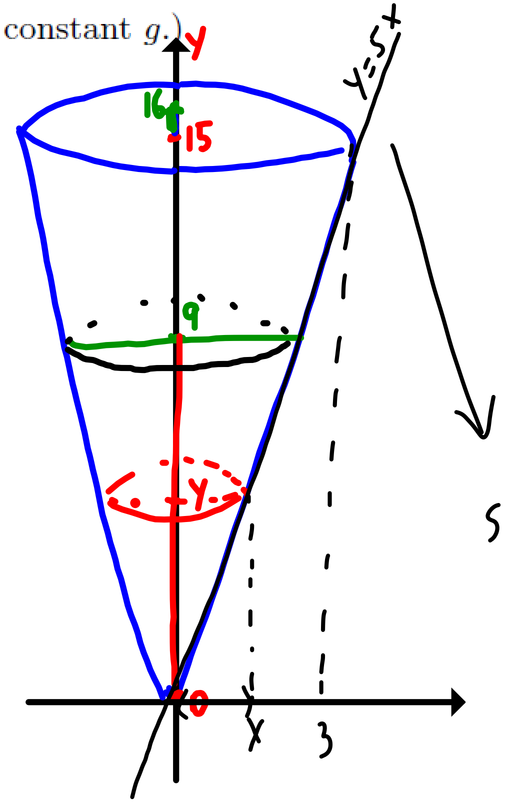
$$A(y) = 20\sqrt{16 - y^2}$$

$$dist(y) = 0.5 - y$$



$$W = \rho g \int_{-3}^{0} 20\sqrt{16 - y^2}\,(0.5 - y)\, dy$$

16. A tank has the shape of an upright circular cone with height $15m$ and radius $3m$. In addition, there is a 1 meter high spout at the top of the cone from which the water exits the tank. If the tank is initially full to a water depth of $9m$, find the work required to pump all of the water out of the spout. Just set up the integral. (Leave your answer in terms of density ρ and the gravitational constant g.)



$$W = \rho g \int_0^9 A(y)\, \text{dist}(y)\, dy$$

$$A(y) = \pi [r(y)]^2 = \pi x^2$$

$$\text{slope} = \frac{15}{3} = 5 \Rightarrow y = 5x \Rightarrow x = \frac{y}{5}$$

$$\Rightarrow A(y) = \pi \left(\frac{y}{5}\right)^2 = \frac{\pi y^2}{25}$$

$$\text{dist}(y) = 16 - y$$

$$W = \rho g \frac{\pi}{25} \int_0^9 y^2 (16 - y)\, dy$$

17. Find the average value of $f(x) = \sin^2 x$ over the interval $\left[0, \frac{\pi}{4}\right]$. $[a, b]$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f_{ave} = \frac{1}{\frac{\pi}{4}-0}\int_0^{\pi/4} \sin^2 x\,dx = \frac{4}{\pi}\int_0^{\pi/4} \frac{1}{2}(1-\cos 2x)\,dx$$

$$= \frac{2}{\pi}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^{\pi/4} = \frac{2}{\pi}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

18. Compute the following integrals:

(a) $I = \int_0^1 (x^2 + 4x - 1)e^x \, dx.$

$u = x^2 + 4x - 1$ $\qquad e^x dx = dv$

$du = 2x + 4$ $\qquad v = e^x$

$$I = uv\Big|_0^1 - \int_0^1 v\,du = e^x(x^2+4x-1)\Big|_0^1 - \int_0^1 (2x+4)e^x dx$$

<u>2nd</u> $u = 2x+4$, $\quad e^x dx = dv$

$du = 2dx \qquad v = e^x$

$$I = 4e+1 - \left[(2x+4)e^x\Big|_0^1 - 2\int_0^1 e^x dx\right]$$

$$I = 4e+1 - [6e - 4 - 2(e-1)] = \boxed{3}$$

(b) $\int \sqrt{x}\,\ln(x)\,dx = \int x^{\frac{1}{2}} \ln(x)\,dx$

$u = \ln x$ $\qquad x^{\frac{1}{2}}\,dx = dv$

$du = \frac{dx}{x}$ $\qquad v = \int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{3/2}$

$$\int = \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}\underbrace{x^{3/2}\frac{dx}{x}}_{x^{\frac{1}{2}}} = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\cdot\frac{2}{3}x^{3/2} + C$$

$$= \frac{2}{3}x\sqrt{x}\ln x - \frac{4}{9}x\sqrt{x} + C$$