Fall 2012 Math 152

Week in Review 5

courtesy: *Dr. Oksana Shatalov*

(covering Sections 8.3,8.4&8.9)

8.3: Trigonometric Substitutions

integral with	substitution	identity
$a^2 - x^2$	$x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$
$ax^2 + bx + c$	complete squares and then do the correct substitution	

Examples

1. Evaluate the given integral:
   (a) $\int \sqrt{1-3x^2}\,\mathrm{d}x$
   (b) $\int \frac{1}{x^2\sqrt{25x^2-9}}\,\mathrm{d}x$
   (c) $\int_0^{4/7} \frac{1}{(49x^2+16)^{3/2}}\,\mathrm{d}x$
2. Use a trigonometric substitution to eliminate the root:
   (a) $\sqrt{(x+1)^2-64}$
   (b) $\sqrt{4(x-5)^2+1}$
3. (a) Use a trigonometric substitution to eliminate the root: $\sqrt{24-2x-x^2}$.
   (b) Evaluate the integral $\int \frac{(x+1)^2}{(24-x^2-2x)^{3/2}}\,\mathrm{d}x$

8.4: Integration Of Rational Functions By Partial Fractions

Key Points

- *Rational function:* $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

- Partial Fraction Decomposition only works on proper rational functions (degree of the numerator is **strictly less** than the degree of denominator.) (Otherwise (for improper rational functions), you must first do **long division**.)
- You have to factor a denominator as much as possible and "break down" the fraction using individual denominators.
- For each **factor** in the denominator use the following table:

Factor in denominator	Term in partial fraction decomposition
linear factor $ax+b$	$\frac{A}{ax+b}$
repeated linear factor $(ax+b)^2$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}$
repeated linear factor $(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
prime quadratic factor ax^2+bx+c, where $b^2-4ac<0$	$\frac{Ax+B}{ax^2+bx+c}$
repeated prime quadratic factor $(ax^2+bx+c)^2$, where $b^2-4ac<0$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}$
repeated prime quadratic factor $(ax^2+bx+c)^k$, where $b^2-4ac<0$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

- Solve for numerators using roots of the denominator and/or matching powers of x. THEN INTEGRATE.

Examples

4. Write out the form of the partial fraction decomposition of the following rational functions. (Do not try to solve)

   (a) $\frac{3x}{(x-1)(3x+12)}$

   (b) $\frac{5x^2}{(x-1)^2(x^2-1)}$

   (c) $\frac{7}{x(x^3+x^2+x)}$

   (d) $\frac{x+5}{(x-3)(x^2+25)^2}$

5. Compute the following integrals.

   (a) $I=\int\frac{x^2+4x}{(x-1)(x-2)(x+3)}\,dx$

   (b) $I=\int\frac{x^2-3x+7}{(x-1)(x^2+1)}\,dx$

   (c) $I=\int\frac{x^4-x^3-12x^2+10}{x^3-4x^2}\,dx$

   (d) $I=\int\frac{1}{(x^2+1)(x^2+x+1)}\,dx$

8.9: Improper Integrals

Key Points

- TYPE I: Infinite Interval and Continuous Integrand:
  $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x$ (likewise with $-\infty$)
- TYPE II: Discontinuous Integrand and Finite Interval:
  $f(x)$ is discontinuous at $x=a$: $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+}\int_t^b f(x)\,\mathrm{d}x$ (likewise with b^-)
- FACT: If $a>0$ then $\displaystyle\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x$ is convergent as $p>1$ and divergent as $p\le 1$.
- **Comparison Theorem:** *Let $f(x)$ and $g(x)$ be continuous and $f(x)\ge g(x)\ge 0$ for $x\ge a$. Then*
  - *if $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ is convergent then $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ is convergent.*
    (Note if $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ is divergent, no conclusion can be drawn about $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$.)
  - *if $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ is divergent then $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ is divergent.*
    (Note if $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ is convergent, no conclusion can be drawn about $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$.)

Examples

6. Compute the following integrals or show they diverge:
   (a) $I = \displaystyle\int_e^\infty \frac{1}{x(\ln(x)^5)}\,\mathrm{d}x$
   (b) $I = \displaystyle\int_{-\infty}^0 (1+x)e^x\,\mathrm{d}x$
   (c) $I = \displaystyle\int_{-\infty}^\infty \frac{5x^4}{(x^5+5)^3}\,\mathrm{d}x$
   (d) $I = \displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-4}}\,\mathrm{d}x$
7. Determine whether the given integrals converge or diverge using the Comparison Theorem.
   (a) $I = \displaystyle\int_0^\infty \frac{1}{x^{2012}+e^{2012x}}\,\mathrm{d}x$
   (b) $I = \displaystyle\int_5^\infty \frac{x^2}{x^{5/2}-x}\,\mathrm{d}x$
   (c) $I = \displaystyle\int_{10}^\infty \frac{\sin^4(7x)}{x^7}\,\mathrm{d}x$