Fall 2012 Math 152

Week in Review 7

courtesy: *Oksana Shatalov*

(covering Section10.2 & Exam 2 Review)

10.2: Series

Key Points

- Infinite series $\sum_{n=1}^{\infty} a_n \quad$ ($n = 1$ for convenience, it can be anything).
- Partial sums: $s_N = \sum_{n=1}^{N} a_n$. Note $\boxed{s_N = s_{N-1} + a_N}$
- If $\{s_N\}_{N=1}^{\infty}$ is convergent and $\lim_{N\to\infty} s_N = s$ exists as a real number, then the series $\sum_{n=1}^{n} a_n$ is *convergent*. The number s is called the **sum** of the series.

- Series we can sum:
  - Geometric Series $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}, \quad -1 < r < 1$
  - Telescoping Series

    $\sum b_n - b_{n+2}$, $\sum b_n - b_{n+2}$, $\sum b_{n+1} - b_{n-1}$

- THE TEST FOR DIVERGENCE: *If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.*
- **The Test for Divergence cannot be used to prove that a series converges. It can only show a series is divergent.**

Examples

1. Given a series whose partial sums are given by $s_n = (7n+3)/(n+7)$, find the general term a_n of the series and determine if the series converges or diverges. If it converges, find the sum.

$$S_n = \frac{7n+3}{n+7}$$

$$S_n = S_{n-1} + a_n \Rightarrow a_n = S_n - S_{n-1}$$

$$a_n = \frac{7n+3}{n+7} - \frac{7(n-1)+3}{n-1+7}$$

$$\boxed{a_n = \frac{7n+3}{n+7} - \frac{7n-4}{n+6}}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{7n+3}{n+7} = \frac{7}{1} = 7$$ $\Rightarrow$ The series converges and its sum is 7.

2. Find the sum of the following series or show they are divergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{7+5^n}{10^n} = \sum_{n=1}^{\infty} 7\cdot\left(\frac{1}{10}\right)^n + \left(\frac{5}{10}\right)^n = \sum_{n=1}^{\infty} \underbrace{7\cdot\left(\frac{1}{10}\right)^n}_{} + \underbrace{\left(\frac{1}{2}\right)^n}_{}$

geometric series

$$= \sum_{n=1}^{\infty} \underbrace{7\cdot\frac{1}{10}}_{a}\cdot\left(\frac{1}{10}\right)^{n-1} + \underbrace{\frac{1}{2}}_{a}\cdot\left(\frac{1}{2}\right)^{n-1}$$

$r=\frac{1}{10}$ $\qquad r=\frac{1}{2}$

In both cases $-1<r<1 \Rightarrow$ the geom. series converges

Use formula $\frac{a}{1-r}$ to find sum

$$= \frac{7/10}{1-1/10} + \frac{1/2}{1-\frac{1}{2}} = \frac{7}{9} + 1 = \boxed{\frac{16}{9}}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+3)} = \sum_{n=1}^{\infty} \overbrace{\frac{4}{n+1} - \frac{4}{n+3}}^{a_n}$

Telescoping series

$$\frac{8}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3}$$

$$8 = A(n+3) + B(n+1)$$

$n=-1 \Rightarrow 8 = 2A \Rightarrow A = 4$

$n=-3 \Rightarrow 8 = -2B \Rightarrow B = -4$

$$\boxed{\frac{4}{n+1} - \frac{4}{n+3} = a_n}$$

$$S_n = a_1 + a_2 + \dots + a_{n-1} + a_n = \overbrace{\boxed{\frac{4}{2}} - \cancel{\frac{4}{4}}}^{n=1} + \overbrace{\boxed{\frac{4}{3}} - \cancel{\frac{4}{5}}}^{n=2} + \overbrace{\cancel{\frac{4}{4}} - \cancel{\frac{4}{6}}}^{n=3} + \dots$$

$$+ \dots + \underbrace{\cancel{\frac{4}{n}} - \boxed{\frac{4}{n+2}}}_{n \to n-1} + \underbrace{\cancel{\frac{4}{n+1}} - \boxed{\frac{4}{n+3}}}_{n}$$

$$S_n = \underbrace{\frac{4}{2} + \frac{4}{3}}_{2 + \frac{4}{3} = \frac{10}{3}} - \frac{4}{n+2} - \frac{4}{n+3}$$

$n \to \infty$: $\frac{4}{n+2} \to 0$, $\frac{4}{n+3} \to 0$

$$\lim_{n \to \infty} S_n = \boxed{\frac{10}{3}}$$ sum of the series.

The series converges.

3. Write the repeating decimal $0.\overline{27}$ as a fraction.

$$0.\overline{27} = 0.27272727\ldots$$

$$= 0.27 + 0.0027 + 0.000027 + 0.00000027 + \ldots$$

$$= 0.27 + 0.27 \cdot 10^{-2} + 0.27 \cdot 10^{-4} + 0.27 \cdot 10^{-6} + \ldots$$

$\times 10^{-2}$ $\times 10^{-2}$ $\times 10^{-2}$

sum of geometric series

with $a = 0.27$, $r = 10^{-2} = 0.01$

$$0.\overline{27} = \frac{a}{1-r} = \frac{0.27}{1-0.01} = \frac{0.27}{0.99} = \boxed{\frac{27}{99}} = \boxed{\frac{3}{11}}$$

4. Use the test for Divergence to determine whether the series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{3(n^4+3)(n+1)}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^5}{3(n^4+3)(n+1)} = \frac{1}{3}$$

$\Rightarrow$ the series diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$

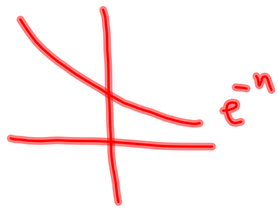
$$\lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0 \Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n\sqrt{n}} = 0$$

no conclusion at the moment

because DT fails

Use another Test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6 - e^{-n}}$



$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{6-\underbrace{e^{-n}}_{\downarrow\ 0}} = \frac{1}{6}$$

The series diverges

Exam 2 Review

1. Evaluate the integral $I = \int (4x^2 - 25)^{-3/2}\, dx = \int \frac{dx}{\left(\sqrt{4x^2-25}\right)^3}$

Trig substitution

$$\sqrt{4x^2-25} = \sqrt{25\left(\frac{4x^2}{25}-1\right)} = \sqrt{5^2\left(\left(\frac{2x}{5}\right)^2-1\right)}$$

$$\frac{2x}{5} = \sec\theta \qquad = \sqrt{5^2(\sec^2\theta - 1)}$$

$$\Downarrow \qquad = \sqrt{5^2\tan^2\theta} = 5\tan\theta$$

$$x = \frac{5}{2}\sec\theta$$

$$dx = \frac{5}{2}\sec\theta\tan\theta\, d\theta$$

$$I = \int \frac{\cancel{\tfrac{5}{2}}\sec\theta\,\cancel{\tan\theta}\,d\theta}{5^{\cancel{3}\,2}\tan^{\cancel{3}\,2}\theta} = \frac{1}{50}\int \frac{\sec\theta\, d\theta}{\tan^2\theta} =$$

$$= \frac{1}{50}\int \frac{d\theta}{\cos\theta \; \frac{\sin^2\theta}{\cos^2\theta}} = \frac{1}{50}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta \qquad \boxed{\begin{aligned} u &= \sin\theta \\ du &= \cos\theta\, d\theta \end{aligned}}$$

$$= \frac{1}{50}\int \frac{du}{u^2} = -\frac{1}{50u} + C$$

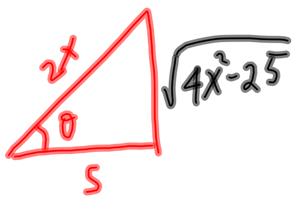
Return to θ

$$= -\frac{1}{50\sin\theta} + C$$

Return to x

$$\sec\theta = \frac{2x}{5}$$

$$\cos\theta = \frac{5}{2x}$$



$$\sin\theta = \frac{\sqrt{4x^2-25}}{2x}$$

$$I = -\frac{1}{\cancel{50}_{25} \cdot \frac{\sqrt{4x^2-25}}{\cancel{2}x}} + C = \boxed{-\frac{x}{25\sqrt{4x^2-25}} + C}$$

2. Determine whether the given integral is convergent or divergent.

(a) $\displaystyle\int_1^{\infty} \frac{4+\cos^4 x}{x}\,\mathrm{d}x$

$$\frac{4+\cos^4 x}{x} \geq \frac{4}{x}$$

$0 \leq \cos^4 x \leq 1$

$\displaystyle\int_1^{\infty} \frac{4}{x}\,dx$ divergent $(p=1)$

By Comp. Theorem the given integral diverges.

(b) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}+e^{4x}}\,dx$

$a,b \geq 0 \Rightarrow a+b \geq b$

$$\frac{1}{a+b} \leq \frac{1}{b}$$

$\cancel{\frac{1}{\sqrt{x}}}$ $\quad \frac{1}{e^{4x}}$

$$\frac{1}{\sqrt{x}+e^{4x}} \leq \frac{1}{e^{4x}}$$

$$\int_0^\infty \frac{1}{e^{4x}}\,dx = \int_0^\infty e^{-4x}\,dx = \lim_{t\to\infty}\int_0^t e^{-4x}\,dx$$

$$= \lim_{t\to\infty}\left(-\frac{1}{4}\right)e^{-4x}\Big|_0^t = -\frac{1}{4}\lim_{t\to\infty}(\underset{\downarrow 0}{e^{-4t}} - e^0) = \boxed{\frac{1}{4}}$$

convergent

Thus by Comparison Theorem
the given integral converges.

3. Evaluate $I = \int_0^{2012} \frac{1}{\sqrt{2012 - x}}\, \mathrm{d}x.$

improper integral of TYPE II
(incontinuous integrand)

$$I = \lim_{t \to 2012^-} \int_0^t \frac{dx}{\sqrt{2012-x}}$$

$u = 2012 - x$

$du = -dx$

$\int \frac{dx}{\sqrt{2012-x}} = -\int u^{-\frac{1}{2}}\, du$

$= -2\sqrt{u} = -2\sqrt{2012-x}$

$$= \lim_{t \to 2012^-} -2\sqrt{2012-x}\,\Big|_0^t$$

$$= -2\left[\sqrt{2012-2012} - \sqrt{2012-0}\right] = \boxed{2\sqrt{2012}}$$

4. The curve $y = \sin x$ for $0 \leq x \leq \pi$ is rotated about the x-axis. Set up, *but don't evaluate* the integral for the area of the resulting surface.

$$SA = 2\pi \int y \, ds = \boxed{2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x} \, dx}$$

$$ds = \sqrt{1 + (y')^2}$$

$$y' = (\sin x)' = \cos x$$

5. Determine if the sequence $\{a_n\}_{n=2}^{\infty}$ is decreasing and bounded:

(a) $a_n = \ln n$

$f(x) = \ln x$

$\{a_n\}$ is not decreasing and unbdd

(b) $a_n = \cos n^2$

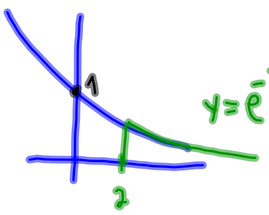
oscillate

not decreasing

bounded, because $-1 \le \cos n^2 \le 1$

(c) $a_n = e^{-n}$

$f(x) = e^{-x}$



$y = e^{-x}$, $x \geq 2$

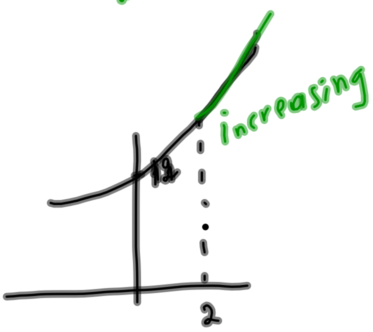
decreasing

bounded

$\left(0 < e^{-x} < 1 \text{ for } x \geq 2 \right)$

(d) $a_n = e^n + 11$

$f(x) = e^x + 11$



not decreasing

not bdd

(e) $a_n = 1 - \frac{1}{n^2}$

$0 < a_n < 1$ bounded

$f(x) = 1 - \frac{1}{x^2} \Rightarrow f'(x) = \frac{1}{x} > 0$ for $x \geq 2$

increasing

not decreasing

6. The curve $y = \frac{1}{2}(e^x + e^{-x})$, $0 \le x \le 1$, is rotated about the x-axis. Find the area of the resulting surface.

$$SA = 2\pi \int y\, ds = \cancel{2}\pi \int_0^1 \cancel{\tfrac{1}{2}}(e^x + e^{-x}) \sqrt{1 + (y')^2}\, dx$$

$$y' = \frac{1}{2}(e^x - e^{-x}) \qquad (a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$y'^2 = \frac{1}{4}(e^x - e^{-x})^2 = \frac{1}{4}\left((e^x)^2 - 2 \cdot e^x \cdot e^{-x} + (e^{-x})^2\right)$$

$$= \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$\sqrt{1 + y'^2} = \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}} = \sqrt{\frac{4 + e^{2x} - 2 + e^{-2x}}{4}}$$

$$= \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}} = \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2} = \frac{e^x + e^{-x}}{2}$$

continued →

$$SA = \pi \int_0^1 (e^x + e^{-x}) \cdot \left(\frac{e^x + e^{-x}}{2}\right) dx$$

$$SA = \frac{\pi}{2} \int_0^1 (e^x + e^{-x})^2 \, dx$$

$$SA = \frac{\pi}{2} \int_0^1 \left(e^{2x} + 2 + e^{-2x}\right) dx$$

$$SA = \frac{\pi}{2} \left(\frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x} \right) \Bigg|_0^1$$

$$= \frac{\pi}{2} \left[\frac{e^2}{2} + 2 - \frac{e^{-2}}{2} - \left(\frac{1}{2} + 0 - \frac{1}{2} \right) \right]$$

$$\boxed{SA = \frac{\pi}{4} \left[e^2 - e^{-2} + 4 \right]}$$

7. Set up, *but don't evaluate* the integral for the length of the curve $x = 2t^2, \quad y = t^3, \quad 0 \le t \le 1$.

$$L = \int ds = \int_0^1 \sqrt{(x')^2 + (y')^2}\, dt = \int_0^1 \sqrt{(4t)^2 + (3t^2)^2}\, dt$$

$$\boxed{L = \int_0^1 \sqrt{16t^2 + 9t^4}\, dt}$$

$0 \le x \le \frac{1}{6}$

8. Find length of the curve $y = \frac{1}{\pi}\ln(\sec(\pi x))$ from the point $(0,0)$ to the point $(\frac{1}{6}, \ln\frac{2}{\sqrt{3}})$.

$$L = \int ds = \int_0^{1/6} \sqrt{1+(y')^2}\, dx$$

$$y' = \frac{1}{\pi}\left(\ln\left(\sec(\pi x)\right)\right)' = \frac{1}{\cancel{\pi}}\,\frac{1}{\cancel{\sec \pi x}}\cdot \cancel{\sec(\pi x)}\tan(\pi x)\cdot\cancel{\pi}$$

$$y' = \tan(\pi x) \Rightarrow 1+(y')^2 = 1+\tan^2(\pi x) = \sec^2(\pi x)$$

$$L = \int_0^{1/6} \sqrt{\sec^2(\pi x)}\, dx = \int_0^{1/6} |\sec(\pi x)|\, dx = \int_0^{1/6} \sec(\pi x)\, dx$$

$$L = \frac{1}{\pi}\ln|\sec(\pi x)+\tan(\pi x)|\Big|_0^{1/6}$$

$$L = \frac{1}{\pi}\left[\ln\left|\sec\frac{\pi}{6}+\tan\frac{\pi}{6}\right| - \underbrace{\ln|\sec 0+\tan 0|}_{\ln 1 = 0}\right]$$

$$L = \frac{1}{\pi}\ln\left|\frac{2}{\sqrt{3}}+\frac{\sqrt{3}}{3}\right|$$

$$\frac{\sqrt{\begin{array}{c|c|c|c|c|c} & 0 & \frac{\pi}{6} & \frac{\pi}{4} & \frac{\pi}{3} & \frac{\pi}{2} \\ \hline \sin & 0 & 1 & 2 & 3 & 4 \\ \hline \cos & 4 & 3 & 2 & 1 & 0 \end{array}}}{2}$$

9. Use a trigonometric substitution to eliminate the root: $\sqrt{24-12x+2x^2}$.

$$\sqrt{24-12x+2x^2} = \sqrt{2(x^2-6x+12)}$$

$(a-b)^2 = a^2-2ab+b^2$

$a=x$ $\quad 6x = 2\cdot x\cdot 3$ $\quad b=3$

$$\sqrt{2(\underbrace{x^2-6x+9}_{(x-3)^2}+3)} = \sqrt{2\left[(x-3)^2+3\right]}$$

$$= \sqrt{2\cdot 3\left[\frac{(x-3)^2}{3}+1\right]}$$

$$= \sqrt{6}\sqrt{\left(\frac{x-3}{\sqrt{3}}\right)^2+1}$$

Trig subst.

$1+\tan^2 x = \sec^2 x$

$$\boxed{\frac{x-3}{\sqrt{3}} = \tan\theta}$$

$$\sqrt{6}\sqrt{\tan^2\theta+1} = \sqrt{6}\sqrt{\sec^2\theta}$$

$$= \sqrt{6}\,|\sec\theta|$$

10. Determine if the sequence converges or diverges. If converges, find its limit.

(a) $\left\{ \dfrac{2012 + (-1)^n}{n^{2012}} \right\}_{n=1}^{\infty}$

$$\frac{2012-1}{n^{2012}} \leq \frac{2012+(-1)^n}{n^{2012}} \leq \frac{2012+1}{n^{2012}}$$

$$\underset{n\to\infty}{\downarrow} \qquad \underset{\text{squeeze Theorem}}{\downarrow} \qquad \underset{n\to\infty}{\downarrow}$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

The sequence converges to zero.

(b) $\left\{\sqrt{\dfrac{7n+6n^3+n^2}{(n+3)(n^2+8)}}\right\}_{n=4}^{\infty} \longrightarrow \boxed{\sqrt{6}}$

$$\| $$

$$\sqrt{\frac{6n^3+\cdots}{n^3+\cdots}}$$ rational function

11. Evaluate the integral $\displaystyle\int \frac{(x-1)^2}{5\sqrt{25-(x-1)^2}}\,dx$.

$$\sqrt{25-(x-1)^2} = \sqrt{25\left(1-\frac{(x-1)^2}{25}\right)} = 5\sqrt{1-\left(\frac{x-1}{5}\right)^2}$$

$$\frac{x-1}{5} = \sin\theta \Rightarrow x = 5\sin\theta + 1$$

$$dx = 5\cos\theta\, d\theta$$

$$= 5\sqrt{1-\sin^2\theta}$$

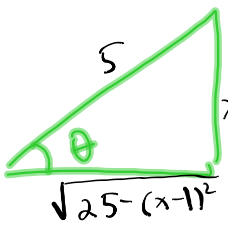
$$= 5\cos\theta$$

$$\int \frac{(x-1)^2\,dx}{5\sqrt{25-(x-1)^2}} = \int \frac{\cancel{5}^2\sin^2\theta \cdot \cancel{5\cos\theta}\,d\theta}{\cancel{5}\cdot\cancel{5}\cancel{\cos\theta}} = 5\int \sin^2\theta\,d\theta$$

$$= 5\int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{5}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{5}{2}\left(\theta - \cos\theta\sin\theta\right) + C$$

Return to x:



$\sin\theta = \frac{x-1}{5}$

$\cos\theta = \frac{\sqrt{}}{5}$

$$\boxed{= \frac{5}{2}\left(\arcsin\frac{x-1}{5} - \frac{\sqrt{25-(x-1)^2}\,(x-1)}{25}\right) + C}$$

12. Compute $S = \sum_{n=1}^{\infty} (\overbrace{\underbrace{e^{1/n}}_{b_n} - \underbrace{e^{1/(n+1)}}_{b_{n+1}}}^{a_n})$.

$$S_n = a_1 + a_2 + \dots + a_{n-1} + a_n = b_1 - \cancel{b_2} + \cancel{b_2} - \cancel{b_3} + \dots + \cancel{b_{n-1}} - \cancel{b_n} + \cancel{b_n} - b_{n+1}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (b_1 - b_{n+1}) = \lim_{n\to\infty} \left(e - e^{\frac{1}{n+1}}\right)$$

$$= \boxed{e - 1}$$

13. Write out the form of the partial fraction decomposition (do not try to solve)

$$\frac{20x^3 + 12x^2 + x}{(x^3 - x)(x^3 + 2x^2 - 3x)(x^2 + x + 1)(x^2 + 9)^2}$$

$x(x^2-1)$ $x(x^2+2x-3)$

$x(x-1)(x+1)$ $x(x+3)(x-1)$

$$\frac{\cancel{x}(20x^2+12x+1)}{x^{\cancel{2}}(x-1)^2(x+1)(x+3)(x^2+x+1)(x^2+9)^2} = \frac{A}{x} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{x+3}$$

L. L.R. L. L. Q.P. Q.P.R.

$$+ \frac{Ex+F}{x^2+x+1} + \frac{G_1x+H_1}{x^2+9} + \frac{G_2x+H_2}{(x^2+9)^2}$$

14. Evaluate the integral $\int \frac{5x^2 + x + 12}{x^3 + 4x}\,\mathrm{d}x$ $= \int \frac{5x^2+x+12}{x(x^2+4)}\,dx$

deg(num) < deg(denom.)

$$\frac{5x^2+x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$5x^2+x+12 = A(x^2+4) + (Bx+C)x$$

$x=0$ $\quad 12 = 4A \Rightarrow \boxed{A=3}$

x^2: $\quad 5 = A+B \Rightarrow 5 = 3+B \Rightarrow \boxed{B=2}$

x: $\quad \boxed{1=C}$

$$\int = \int \frac{A}{x} + \frac{Bx+C}{x^2+4}\,dx = \int \frac{3}{x} + \frac{2x}{x^2+4} + \frac{1}{x^2+4}\,dx$$

$$= 3\ln|x| + \ln(x^2+4) + \frac{1}{2}\arctan\frac{x}{2} + C$$

15. Assuming that the sequence defined recursively by $a_1 = 1$, $\boxed{a_{n+1} = \frac{1}{2}\left(a_n + \frac{16}{a_n}\right)}$ is convergent, find its limit.

Denote $\lim_{n\to\infty} a_n = L > 0$

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = L$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{16}{a_n}\right) = \frac{1}{2}\left(L + \frac{16}{L}\right) = L$$

$$L + \frac{16}{L} = 2L$$

$$\frac{L^2+16}{L} = 2L \Rightarrow L^2 + 16 = 2L^2$$

$$L^2 = 16$$

$$L = \pm 4$$

$$\boxed{L = 4}$$

16. For what values of x the series $\sum_{n=0}^{\infty}(4x-3)^{n+3}$ converges? What is the sum of the series?

Note that the series is geometric

$$\sum_{n=0}^{\infty}(4x-3)^{n+3} = \sum_{n=0}^{\infty}\underbrace{(4x-3)^3}_{a}\underbrace{(4x-3)^n}_{r^n}$$

$r = 4x-3$

Geom. series converges $\Leftrightarrow$ $|r|<1$

$-1<r<1$

$-1<4x-3<1$

$-1+3<4x<1+3$

$2<4x<4$

$\frac{1}{2}<x<1$

For $\boxed{\frac{1}{2}<x<1}$ we have

$$\sum_{n=0}^{\infty}(4x-3)^{n+3} = \frac{a}{1-r} = \frac{(4x-3)^3}{1-(4x-3)} = \boxed{\frac{(4x-3)^3}{4-4x}}$$