Fall 2012 Math 152

Week in Review 8

courtesy: *Oksana Shatalov*

(covering Section 10.3& 10.4)

10.3 : The Integral and Comparison Tests; Estimating Sums

Key Points

THE TEST FOR DIVERGENCE: *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum a_n$ *is divergent.*	If $\lim_{n\to\infty} a_n = 0$ then the series may or may not converge.
THE INTEGRAL TEST *Let* $\sum a_n$ *be a* **positive** *series. If* f *is a continuous and decreasing function on* $[a, \infty)$ *such that* $a_n = f(n)$ *for all* $n \geq a$ *then* $\sum a_n$ *and* $\int_a^\infty f(x)\,\mathrm{d}x$ *both converge or both diverge.*	Apply to positive series only when $f(x)$ is easy to integrate.
THE COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with* **nonnegative** *terms and* $a_n \leq b_n$ *for all* n. • *If* $\sum b_n$ *is convergent then* $\sum a_n$ *is also convergent.* • *If* $\sum a_n$ *is divergent then* $\sum b_n$ *is also divergent.*	• It applies to series with nonnegative terms only. • Try it as a last resort (other tests are often easier to apply). • It requires some skills in chosing a series for comparison.
LIMIT COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with* **positive** *terms . If* $$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$ *where* c *is a finite number and* $c > 0$, *then either both series converge or both diverge.*	• It applies to positive series only. • It requires less skills to choose series for comparison than in Comparison test.

- *FACT:* **The** p**-series,** $\sum_{n=1}^{\infty} \frac{1}{n^p}$**, converges if** $p > 1$ **and diverges if** $p \leq 1$**.**(by Integral Tests)
- REMAINDER ESTIMATE FOR THE INTEGRAL TEST

  If $\sum a_n$ *converges by the Integral Test and* $R_n = s - s_n$, *then*

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq R_n \leq \int_n^{\infty} f(x)\,\mathrm{d}x$$

Examples

1. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$ is convergent or divergent.

2. Find the values of p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is divergent.

3. Determine if the following series is convergent or divergent:

   (a) $\sum_{n=1}^{\infty} \frac{0.99}{n^{0.99}}$

   (b) $\sum_{n=1}^{\infty} \frac{1.01}{n^{1.01}}$

   (c) $\sum_{n=1}^{\infty} \frac{2012}{\sqrt[7]{n^5}\sqrt[3]{8n}}$

   (d) $\sum_{n=1}^{\infty} \frac{n^2+12}{\sqrt{n^6+6}}$

   (e) $\sum_{n=1}^{\infty} \frac{n^{10}}{n^{15}+n^{11}-7}$

   (f) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^7}\right)$

   (g) $\sum_{n=1}^{\infty} \frac{5n^5+e^{-5n}}{6n^6-e^{-6n}}$

4. Find the values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)n^p}$ is convergent.

5. (a) If $\sum_{n=1}^{1000} \frac{1}{n^6}$ is used to approximate $\sum_{n=1}^{\infty} \frac{1}{n^6}$, find an upper bound on the error using the Integral Test.

   (b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^6}$ correct to 11 decimal places.

6. Given the series $\sum_{n=1}^{\infty} n^3 e^{-n^4}$.

   (a) Show that the series converges.

   (b) Find an upper bound for the error approximating this series by its 5th partial sum s_5.

10.4 : Other Convergence Tests

Key Points

ALTERNATING SERIES TEST: *If* $b_n > 0$, $\lim\limits_{n\to\infty} b_n = 0$ *and the sequence* $\{b_n\}$ *is decreasing then the series* $\sum(-1)^n b_n$ *is convergent.*	It applies only to alternating series.
RATIO TEST For a series $\sum a_n$ with nonzero terms define $L = \lim\limits_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\|$. • If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.) • If $L > 1$ then the series is divergent. • If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).	• Try it when a_n involves factorials or n-th powers. • The series need not have positive terms and need not be alternating to use it. • Absolute convergence implies convergence.

The Alternating Series Theorem. *If* $\sum\limits_{n=1}^{\infty}(-1)^n b_n$ *is a convergent alternating series and you used a partial sum* s_n *to approximate the sum* s *(i.e.* $s \approx s_n$*) then* $|R_n| \leq b_{n+1}$.

Examples

7. Determine whether the following series converges absolutely, converges but not absolutely, or diverges.

   (a) $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^p}$, where p is a real parameter.

   (b) $\sum\limits_{n=2}^{\infty}\frac{(-1)^n}{n\sqrt[4]{\ln n}}$

   (c) $\sum\limits_{n=1}^{\infty}\frac{(-9)^n}{(n+1)!}$

   (d) $\sum\limits_{n=5}^{\infty}\frac{(-1)^{n-1}7^{n-1}}{4^n}$

   (e) $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}(n!)^2}{((2n)!)^2}$

   (f) $\sum\limits_{n=1}^{\infty}\frac{n\cos(n\pi)}{n^2+n+1}$

   (g) $\sum\limits_{n=1}^{\infty}\frac{n^n}{(n!)^2}$

   (h) $\sum\limits_{n=1}^{\infty}\frac{5^n}{\ln(n+1)}$

8. Which of the following statements is TRUE?

   (a) If $a_n > 0$ for $n \geq 1$ and $\sum_{n=1}^{\infty}(-1)^n a_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

   (b) If $a_n > 0$ for $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

   (c) If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

   (d) If $a_n > 0$ for $n \geq 1$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{e}{2}$ then $\sum_{n=1}^{\infty} a_n$ converges.

9. Given the series $\sum_{n=1}^{\infty}(-1)^{n+1} n^3 e^{-n^4}$.

   (a) Show that the series converges.

   (b) Find an upper bound for the error approximating this series by its 5th partial sum s_5.