Fall 2012 Math 152

Week in Review 8

courtesy: *Oksana Shatalov*

(covering Section 10.3& 10.4)

10.3 : The Integral and Comparison Tests; Estimating Sums

DT THE TEST FOR DIVERGENCE: *If* $\lim\limits_{n\to\infty} a_n$ *does not exist or if* $\lim\limits_{n\to\infty} a_n \neq 0$*, then the series* $\sum a_n$ *is divergent.*	If $\lim\limits_{n\to\infty} a_n = 0$ then the series may or may not converge.
IT THE INTEGRAL TEST *Let* $\sum a_n$ *be a* **positive** *series. If* f *is a continuous and decreasing function on* $[a, \infty)$ *such that* $a_n = f(n)$ *for all* $n \geq a$ *then* $\sum a_n$ *and* $\int_a^\infty f(x)\,\mathrm{d}x$ *both converge or both diverge.*	Apply to positive series only when $f(x)$ is easy to integrate.
CT THE COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with* **nonnegative** *terms and* $a_n \leq b_n$ *for all* n*.* • *If* $\sum b_n$ *is convergent then* $\sum a_n$ *is also convergent.* • *If* $\sum a_n$ *is divergent then* $\sum b_n$ *is also divergent.*	• It applies to series with nonnegative terms only. • Try it as a last resort (other tests are often easier to apply). • It requires some skills in chosing a series for comparison.
LCT LIMIT COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with* **positive** *terms . If* $$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$ *where* c *is a finite number and* $c > 0$*, then either both series converge or both diverge.*	• It applies to positive series only. • It requires less skills to choose series for comparison than in Comparison test.

1. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$ is convergent or divergent.

positive

Divergence Test $\lim_{n\to\infty} \frac{1}{n(\ln n)^4} = 0$ DT fails

Apply Integral Test because

$f(x) = \frac{1}{x(\ln x)^4}$ is easy to integrate

$u = \ln x$

$du = \frac{dx}{x}$

$$\int f(x)dx = \int \frac{dx}{x(\ln x)^4} = \int \frac{du}{u^4} = \int u^{-4}du$$

$$= \frac{u^{-4+1}}{-4+1} = -\frac{1}{3u^3} = -\frac{1}{3(\ln x)^3}$$

$$\int_2^\infty f(x)dx = \lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^4} = \lim_{t\to\infty} -\frac{1}{3(\ln x)^3}\Big|_2^t$$

$$= -\frac{1}{3}\lim_{t\to\infty} \underbrace{\frac{1}{(\ln t)^3}}_{\downarrow\, 0} - \frac{1}{(\ln 2)^3} = \frac{1}{3(\ln 2)^3}$$

converges

$\Downarrow$

the series converges (by Integral Test)

Note that

$f(x) = \frac{1}{x(\ln x)^4}$ is continuous on $[2,\infty)$

and also decreasing:

$$f'(x) = \left(\frac{1}{x(\ln x)^4}\right)' = \frac{(\ln x)^4 + x\cdot 4(\ln x)^3\cdot\frac{1}{x}}{-(x(\ln x)^4)^2} < 0$$

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2. Find the values of p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is divergent.

Define $f(x) = \frac{1}{x(\ln x)^p}$, x in $[2, \infty)$

$f(x)$ is positive, continuous, decreasing $\to$ $f'(x) = -\frac{(\ln x)^p + x\, p(\ln x)^{p-1}\cdot\frac{1}{x}}{(x(\ln x)^p)^2} < 0$

$$\int f(x)\,dx = \int \frac{dx}{x(\ln x)^p} \overset{u=\ln x}{=} \int \frac{du}{u^p} = \int u^{-p}\,du = \frac{u^{-p+1}}{-p+1}$$

$$= \frac{u^{1-p}}{1-p} = \frac{(\ln x)^{1-p}}{1-p}, \quad p \neq 1$$

Apply Integral Test:

$$\int_2^\infty f(x)\,dx = \lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^p} = \lim_{t\to\infty} \left.\frac{(\ln x)^{1-p}}{1-p}\right|_2^t$$

$$= \frac{1}{1-p} \underline{\lim_{t\to\infty} (\ln t)^{1-p}} - (\ln 2)^{1-p}$$

The integral converges if and only if $1-p<0$

$\boxed{1 < p}$

It remains to consider case $p=1$:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \qquad \int_2^\infty \frac{dx}{x\ln x} \overset{u=\ln x}{=} \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u} = \lim_{t\to\infty} \left.\ln u\right|_{\ln 2}^{\ln t} =$$

$$= \lim_{t\to\infty} \ln(\ln t) - \ln(\ln 2)$$

divergent

$\Downarrow$

The series diverges

Answer: The series diverges if $p \leq 1$.

☆ *FACT*: **The p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges if $p > 1$ and diverges if $p \leq 1$.** (by Integral Tests)

3. Determine if the following series is convergent or divergent:

p-series

(a) $\sum_{n=1}^{\infty} \frac{0.99}{n^{0.99}}$ $p = 0.99 < 1 \Rightarrow$ divergent

(b) $\sum_{n=1}^{\infty} \frac{1.01}{n^{1.01}}$ $\Rightarrow p = 1.01 > 1 \Rightarrow$ convergent

(c) $\sum_{n=1}^{\infty} \frac{2012}{\sqrt[7]{n^5}\sqrt[3]{8n}} = \sum_{n=1}^{\infty} \frac{2012}{n^{\frac{5}{7}} \cdot 2 n^{\frac{1}{3}}} \Rightarrow p = \frac{5}{7} + \frac{1}{3} = \frac{15+7}{21} = \frac{22}{21} > 1 \Rightarrow$

$\Rightarrow$ convergent

3. Determine if the following series is convergent or divergent:

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\overset{\text{positive}}{n^2+12}}{\sqrt{n^6+6}} \sim \sum \frac{n^2}{\sqrt{n^6}} = \sum \frac{n^2}{n^3} = \sum \frac{1}{n}$ positive, divergent $p=1$

Apply LCT $\lim_{n\to\infty} a_n \cdot \frac{1}{b_n}$

$$\lim_{n\to\infty} \frac{n^2+12}{\sqrt{n^6+6}} \cdot n = \lim_{n\to\infty} \frac{n^3+12n}{\sqrt{n^6+6}} = \lim_{n\to\infty} \sqrt{\frac{(n^3+12n)^2}{n^6+6}} = 1 > 0$$

$\Rightarrow$ the series diverges by LCT

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{n^{15}+n^{11}-7} = -\frac{1}{5} + \underbrace{\sum_{n=2}^{\infty} \frac{n^{10}}{n^{15}+n^{11}-7}}$ positive

$$\sum \frac{n^{10}}{n^{15}} = \sum_{n=2}^{\infty} \frac{1}{n^5}$$

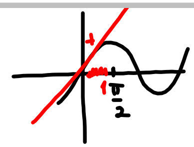
convergent $p=5>1$

Apply LCT $\lim_{n\to\infty} a_n \cdot \frac{1}{b_n}$

$$\lim_{n\to\infty} \frac{n^{10}}{n^{15}+n^{11}-7} \cdot n^5 = \lim_{n\to\infty} \frac{n^{15}}{n^{15}+n^{11}-7} = 1 > 0 \Rightarrow$$

$\Rightarrow$ the series converges by Comparison with $\sum \frac{1}{n^5}$

(f) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\sin\left(\frac{1}{n^7}\right)}$ positive for $n \geq 1$



$\sin x \approx x$ near $x=0$

Try to compare the series to $\displaystyle\sum \underbrace{\frac{1}{n^7}}_{\text{convergent}}$

LCT $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n^7}}{\frac{1}{n^7}} \underset{\substack{x\to 0\\ n\to\infty}}{\overset{x=\frac{1}{n^7}}{=\!=}} \lim_{x\to 0} \frac{\sin x}{x} = 1 > 0$

$\Rightarrow$ The series converges by LCT

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{5n^5 + e^{-5n}}{6n^6 - e^{-6n}}$ $\sim \displaystyle\sum \frac{5n^5}{6n^6} = \underbrace{\sum \frac{5}{6n}}_{\substack{\text{divergent}\\ p=1}}$

$\Downarrow$ estimate general term from below

Apply Comparison Test

$$\frac{\overset{>0}{5n^5} + \overset{>0}{e^{-5n}}}{6n^6 - e^{-6n}} \geq \frac{5n^5}{6n^6 - e^{-6n}} \geq \frac{5n^5}{6n^6} = \frac{5}{6n}$$

$a > b > 0$

$0 < a - b < a$

$\frac{1}{a-b} > \frac{1}{a}$

$6n^6 > 0$

$e^{-6n} > 0$

$6n^6 > e^{-6n}$

$\Rightarrow \frac{1}{6n^6 - e^{-6n}} > \frac{1}{6n^6}$

Conclusion: The series diverges by Comparison to $\sum \frac{5}{6n}$.

4. Find the values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)n^p}$ is convergent.

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)n^p} = \sum_{n=1}^{\infty} \frac{1}{n^{p+1}+n^p} \sim \underbrace{\sum \frac{1}{n^{p+1}}}_{\text{converges } p+1>1}= \sum b_n$$

converges $p+1>1$
$p>0$

diverges $p+1 \le 1$
$p \le 0$

Apply Limit Comparison Test:

$$\lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = \lim_{n\to\infty} \frac{1}{n^{p+1}+n^p} \cdot n^{p+1} = \lim_{n\to\infty} \frac{n^{p+1}/n^{p+1}}{\frac{n^{p+1}+n^p}{n^{p+1}}} =$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 > 0$$

$\Rightarrow$ the original series and $\sum \frac{1}{n^{p+1}}$ either both converge or both diverge.

$$\boxed{\text{Finally, the given series converges if } p>0}$$

- REMAINDER ESTIMATE FOR THE INTEGRAL TEST

If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

5. (a) If $\underbrace{\sum_{n=1}^{1000} \frac{1}{n^6}}_{= S_{1000}\text{ partial sum}}$ is used to approximate $\sum_{n=1}^{\infty} \frac{1}{n^6}$, find an upper bound on the error using the Integral Test.

$\sum_{n=1}^{\infty} \frac{1}{n^6} \approx S_{1000}$

Find upper bound for R_{1000}

$f(x) = \frac{1}{x^6}$

$$R_{1000} \le \int_{1000}^{\infty} \frac{1}{x^6}dx = \lim_{t\to\infty} \int_{1000}^{t} x^{-6}dx = \lim_{t\to\infty} -\frac{1}{5x^5}\Big|_{10^3}^{t}$$

$$= -\frac{1}{5} \underbrace{\lim_{t\to\infty} \frac{1}{t^5}}_{=0} - \frac{1}{(10^3)^5} = -\frac{1}{5}\left(\frac{1}{10^{15}}\right) = \boxed{0.2\cdot 10^{-15}}$$

(b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^6}$ correct to 11 decimal places.

Approximate $\sum_{n=1}^{\infty} \frac{1}{n^6} \approx S_n$

where n is such that $R_n \le 10^{-11}$

We know that $R_n \le \int_n^{\infty} \frac{dx}{x^6} = \lim_{t\to\infty} \int_n^t \frac{dx}{x^6} = -\frac{1}{5}\lim_{t\to\infty} \frac{1}{x^5}\Big|_n^t$

$$R_n \le -\frac{1}{5} \underbrace{\lim_{t\to\infty} \frac{1}{t^5}}_{=0} - \frac{1}{n^5} = \underbrace{\frac{1}{5n^5} \le 10^{-11}}_{\text{Solve that inequality}}$$

$$\frac{1}{5n^5} \le \frac{1}{10^{11}}$$

$$5n^5 \ge 10^{11} = 10\cdot 10^{10}$$

$$n^5 \ge 2\cdot 10^{10}$$

$$n \ge \sqrt[5]{2\cdot 10^{10}}$$

$$n \ge \underbrace{\sqrt[5]{2}}_{\approx 1.149} \cdot \underbrace{\sqrt[5]{10^{10}}}_{=100} \approx 114.9$$

Since n is whole number we get $n = 115$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} \approx \sum_{n=1}^{115} \frac{1}{n^6} = 1 + \frac{1}{2^6} + \dots + \frac{1}{115^6}$$

6. Given the series $\boxed{\sum_{n=1}^{\infty} n^3 e^{-n^4}. \text{ positive series}}$ $f(x) = x^3 e^{-x^4}$ cont.

(a) Show that the series converges.

$$f'(x) = \left(x^3 e^{-x^4}\right)' = 3x^2 e^{-x^4} + x^3 \cdot (-4x^3)\, e^{-x^4}$$

$$= \underbrace{x^2 e^{-x^4}}_{>0} \underbrace{(3 - 4x^4)}_{<0} < 0 \quad \text{for } x \geq 1$$

$\Rightarrow f(x)$ decreasing

Easy to integrate:

$u = -x^4 \Rightarrow du = -4x^3 dx$

$$\int f(x)dx = \int x^3 e^{-x^4} dx = \int -\frac{e^u}{4} du = -\frac{1}{4} e^u = -\frac{1}{4} e^{-x^4}$$

Apply Integral Test

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^t x^3 e^{-x^4} dx = -\frac{1}{4} \lim_{t\to\infty} e^{-x^4}\Big|_1^t =$$

$$= -\frac{1}{4} \underbrace{\lim_{t\to\infty} \left(e^{-t^4}\right.}_{=0} \left. - e^{-1}\right) = \frac{1}{4e}$$

$\Rightarrow$ the integral converges

$\Rightarrow$ the series converges by Integral test

(b) Find an upper bound for the error approximating this series by its 5th partial sum s_5.

$$R_5 \leq \int_5^{\infty} x^3 e^{-x^4} dx \overset{\text{by (a)}}{=} -\frac{1}{4} \lim_{t\to\infty} \left(e^{-t^4} - e^{-5^4}\right) =$$

$e^{-t^4} \downarrow 0$

$$= \boxed{\frac{1}{4e^{625}}}$$

10.4 : Other Convergence Tests

Key Points

AST ALTERNATING SERIES TEST: *If* $b_n > 0$, $\lim_{n\to\infty} b_n = 0$ *and the sequence* $\{b_n\}$ *is decreasing then the series* $\sum(-1)^n b_n$ *is convergent.*	It applies only to alternating series.
RT RATIO TEST For a series $\sum a_n$ with nonzero terms define $L = \lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\|$. • If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.) • If $L > 1$ then the series is divergent. • If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).	• Try it when a_n involves factorials or n-th powers. • The series need not have positive terms and need not be alternating to use it. • Absolute convergence implies convergence.

7. Determine whether the following series converges absolutely, converges but not absolutely, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$, where p is a real parameter.

$\lim_{n\to\infty} \frac{1}{n^p} = \infty$ or DNE if $\boxed{p \le 0}$ $\Rightarrow$ By Divergence Test the series diverges

$\boxed{p>0} \Rightarrow \lim_{n\to\infty} \frac{1}{n^p} = 0$, $\left\{\frac{1}{n^p}\right\}$ decreasing $\Bigg\}$ the series converges by AST

Check abs. conv. $\sum \left|\frac{(-1)^n}{n^p}\right| = \sum \frac{1}{n^p}$ converges if $p>1$

abs. convergent when $p>1$

convergent, but not absolutely, $0<p\le 1$ (converges conditionally)

divergent when $p \le 0$

(b) $\underbrace{\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt[4]{\ln n}}}_{AS}$

$\underbrace{\text{By Example 2 the series } \sum \frac{1}{n\sqrt[4]{\ln n}} \text{ diverges } \quad p = \frac{1}{4} \leq 1}_{\Downarrow}$

no abs. convergence.

$$\left.\begin{array}{l} \lim_{n\to\infty} \frac{1}{n\sqrt[4]{\ln n}} = 0 \\ \left\{ \frac{1}{n\sqrt[4]{\ln n}} \right\} \text{ decreasing (Check it!)} \end{array}\right\} \Rightarrow \text{by AST the series converges.}$$

Remark It follows that the series converges conditionally

(c) $\sum_{n=1}^{\infty} \frac{(-9)^n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^n 9^n}{(n+1)!}$

Use Ratio Test

$|a_n| = \frac{9^n}{(n+1)!}$; $|a_{n+1}| = \frac{9^{n+1}}{(n+1+1)!} = \frac{9^{n+1}}{(n+2)!}$

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{9^{\cancel{n+1}}}{\underbrace{(n+2)!}_{\cancel{(n+1)!}(n+2)}} \cdot \frac{\cancel{(n+1)!}}{\cancel{9^n}} = 9 \lim_{n\to\infty} \frac{1}{n+2} = 0 < 1$$

By Ratio Test the series converges absolutely

(d) $\sum_{n=5}^{\infty} \frac{(-1)^{n-1} 7^{n-1}}{4^n} = \sum_{n=5}^{\infty} \frac{(-1)^{n-1}}{4} \cdot \left(\frac{7}{4}\right)^{n-1}$

$4^n = 4 \cdot 4^{n-1}$

geometric series

$r = \frac{7}{4} > 1$

$\boxed{\text{divergent.}}$

Note that AST fails here:

$$\lim_{n\to\infty} \frac{7^{n-1}}{4^n} = \lim_{n\to\infty} \frac{1}{4} \cdot \left(\frac{7}{4}\right)^{n-1} = \infty$$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{((2n)!)^2}$

Note $(2n)! \neq 2n!$

Ratio Test

$$|a_n| = \left[\frac{n!}{(2n)!}\right]^2 \qquad |a_{n+1}| = \left[\frac{(n+1)!}{(2(n+1))!}\right]^2 = \left[\frac{(n+1)!}{(2n+2)!}\right]^2$$

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \left[\frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!}\right]^2 =$$

$$= \lim_{n\to\infty} \left[\frac{\cancel{n!}\,(n+1)\;\cancel{(2n)!}}{\cancel{(2n)!}\,(2n+1)(2n+2)\;\cancel{n!}}\right]^2 = \lim_{n\to\infty} \left[\frac{n+1}{(2n+1)(2n+2)}\right]^2 = 0 < 1$$

$\Rightarrow$ the series converges absolutely

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{n^2+n+1}$

$\cos \pi n = \begin{cases} 1 & , n \text{ is even} \\ -1 & , n \text{ is odd} \end{cases} = (-1)^n$

$$\| \qquad \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+n+1} \quad \text{alternating series}$$

$$\lim_{n\to\infty} \frac{n}{n^2+n+1} = 0$$

$\left\{\frac{n}{n^2+n+1}\right\}$ decreasing because if $f(x) = \frac{x}{x^2+x+1}$

then $f'(x) = \frac{1-2x^2}{(x^2+x+1)^2} < 0$

for $x \geq 1$

By AST

the series converges

For abs. convergence consider

$$\sum_{n=1}^{\infty} \frac{n}{n^2+n+1} \sim \sum \frac{n}{n^2} = \sum \frac{1}{n} \text{ divergent}$$

Use LCT

$$\lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = \lim_{n\to\infty} \frac{n}{n^2+n+1} \cdot n = 1 > 0$$

$\Rightarrow$

$\Rightarrow$ the series of absolute values divergent $\Rightarrow$

$\Rightarrow$ no absolute convergence

Note The series converges conditionally

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{\ln(n+1)}$

postponed to 11/26
(Night before drill)

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8. Which of the following statements is TRUE?

(a) If $a_n > 0$ for $n \geq 1$ and $\sum_{n=1}^{\infty}(-1)^n a_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $a_n > 0$ for $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

(c) If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

(d) If $a_n > 0$ for $n \geq 1$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{e}{2}$ then $\sum_{n=1}^{\infty} a_n$ converges.

The Alternating Series Theorem. *If* $\sum_{n=1}^{\infty}(-1)^n b_n$ *is a convergent alternating series and you used a partial sum* s_n *to approximate the sum* s *(i.e.* $s \approx s_n$*) then* $|R_n| \le b_{n+1}$.

9. Given the series $\sum_{n=1}^{\infty}(-1)^{n+1}\underbrace{n^3e^{-n^4}}$

(a) Show that the series converges.

By Ex 6a the series $\sum n^3 e^{-n^4}$ converges $\Rightarrow$

$\Rightarrow$ the given series converges absolutely $\Rightarrow$

$\Rightarrow$ converges

Another proof: Use Alternating series Test:

$$\lim_{n\to\infty} n^3e^{-n^4} = \lim_{n\to\infty}\frac{n^3}{e^{n^4}} \underset{\text{L'Hospital's Rule}}{\overset{\frac{\infty}{\infty}}{=}} \lim_{n\to\infty}\frac{3\cancel{n^2}}{4n^{\cancel{3}}e^{n^4}} = 0$$

Also the sequence $\{n^3e^{-n^4}\}$ is decreasing because function $f(x) = x^3e^{-x^4}$ is decreasing on $[1, \infty)$:

$$f'(x) = 3x^2e^{-x^4} + x^3(-4x^3)e^{-x^4} = x^2e^{-x^4}(3-4x^4) < 0$$

Thus, the given series converges by AST.

(b) Find an upper bound for the error approximating this series by its 5th partial sum s_5.

By AS theorem $|R_n| \le b_{n+1}$

$n=5$ and $b_n = n^3e^{-n^4}$ $\Bigg\} \Rightarrow$

$$\Rightarrow |R_5| \le b_6 = \boxed{6^3e^{-6^4}} = \frac{6^3}{e^{6^4}}$$