Fall 2012 Math 152

Week in Review 11

courtesy: *Oksana Shatalov*

(covering Sections 11.1-11.2 & Sample Test 3)

11.1: Three-dimensional Coordinate System

Key Points

- The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- Equation of a sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ (completing the square)

Examples

1. Graph the following regions:
   (a) $x = 5$ in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$; **(b)** $x^2 + y^2 - 1 = 0$ in $\mathbb{R}^2, \mathbb{R}^3$.

2. Given the sphere $(x - 1)^2 + (y + 4)^2 + (z - 2)^2 = 16$.

   (a) What is the intersection of the sphere with the yz-plane.

   (b) Find the distance from the point $(1, -2, 3)$ to the center of the sphere.

3. What is the intersection of the surface $x^2 + y^2 = 49$ with the xy-plane.

4. Determine the radius and the center of the sphere given by the equation

$$x^2 + y^2 + z^2 + 2y + z - 1 = 0.$$

11.2: Vectors and the Dot Product in Three Dimensions

Key Points

- The vector $\mathbf{a}$ from the point $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

- **The magnitude or length** of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

- `Unit vector:` $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$

- **Dot Product**:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\theta = a_1b_1 + a_2b_2 + a_3b_3,$$

  where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \le \theta \le \pi$.

- Scalar projection of vector $\mathbf{b}$ onto vector $\mathbf{a}$: $\mathtt{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

- Vector projection of vector $\mathbf{b}$ onto vector $\mathbf{a}$: $\mathtt{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$

Examples

5. Given the triangle with vertices $A(2,4,5)$, $B(3,5,3)$, and $C(2,8,-3)$.

   (a) Find the cosine of the angle at B.

   (b) Compute $\text{comp}_{\vec{AB}}\vec{BC}$.

   (c) Compute $\text{proj}_{\vec{AB}}\vec{BC}$.

6. Find a unit vector in the direction $\mathbf{b}-\mathbf{a}$ where $\mathbf{a}=\langle 3,-7,0\rangle$ and $\mathbf{b}=\langle 1,-6,-2\rangle$.

7. The line L_1 passes througn the points $M(3,5,3)$ and $N(2,8,-3)$, and the line L_2 passes througn the points $P(-1,-1,1)$ and $Q(12,0,-1)$. Are these lines orthogonal?

Sample Test 3

8. Which of the following series converges absolutely?

   (a) $\sum_{n=1}^{\infty}(-1)^{n+5}$

   (b) $\sum_{n=1}^{\infty}\frac{\sin(\pi^3 n^2)}{n^2\sqrt{n}}$

   (c) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[4]{n}}$

   (d) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln n}$

   (e) $\sum_{n=1}^{\infty}\frac{n^n}{(n!)^2}$

   (f) $\sum_{n=1}^{\infty}\frac{5^n}{\ln(n+1)}$

9. Suppose that the power series $\sum_{n=1}^{\infty}c_n(x-4)^n$ has the radius of convergence 4. Consider the following pair of series:

$$(I)\quad \sum_{n=1}^{\infty}c_n 5^n \qquad (II)\quad \sum_{n=1}^{\infty}c_n 3^n.$$

   Which of the following statements is true?

   (a) (I) is convergent, (II)is divergent

   (b) Neither series is convergent

   (c) Bith series are convergent

   (d) (I) is divergent, (II) is convergent

   (e) no conclusion can be drawn about either series.

10. Show that the series $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ converges. Then find un upper bound on the error in using s_{10} to approximate the series. (Note that $\ln 2 > 1/2$.)

11. If we represent $\dfrac{x^2}{4+9x^2}$ as a power series centered at $a = 0$, what is the associated radius of convergence?

12. Find the radius and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n(3x-1)^n}{n}$.

13. Which of the following statements is TRUE?

    (a) If $a_n > 0$ for $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ converges then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

    (b) If $a_n > 0$ for $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges then $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

    (c) If $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

    (d) If $a_n > 0$ for $n \geq 1$ and $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{e}{2}$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

14. Find a Maclaurin series representation for $\dfrac{e^x - 1 - x}{x^2}$.

15. (a) Find a Maclaurin series representation for $f(x) = \sin\left(\dfrac{x^2}{4}\right)$

    (b) Write $\displaystyle\int_0^1 \sin\left(\frac{x^2}{4}\right) \mathrm{d}x$ as an infinite series.

    (c) Using the series found in the previous part, find s_2, the second partial sum of the series and give an upper bound on the error $|s - s_2|$.

16. Let $f(x) = e^{5-x}$.

    (a) Give the fourth degree Taylor polynomial for $f(x)$ centered around $a = 5$.

    (b) Use Taylor's inequality to give a bound on the error when using the fourth degree Taylor polynomial for $f(x)$ to estimate $f(x)$ on the interval $[3, 6]$.

17. Find a Maclaurin series of $f(x) = \ln(2 - x)$ and the associated radius of convergence.

18. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 3^n}$ converges to s. Use the Alternating Series Theorem to estimate $|s - s_6|$.