Section 1.1: Vectors

Quantities that we measure that have magnitude but not direction are called **scalars**.

DEFINITION 1. *A **vector** is a quantity that has both magnitude and direction.*

Vectors are drawn as directed line segments and typically are denoted by a boldfaced character or a character with and arrow on it (i.e. $\mathbf{a}$ or $\overrightarrow{a}$).

A *2-dimensional vector* is an ordered pair $\mathbf{a} = \langle a_1, a_2 \rangle$. The numbers a_1 and a_2 are called the **components** of the vector $\mathbf{a}$.

DEFINITION 2. *Given the points $A(a_1, a_2)$ and $B(b_1, b_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is*

$$\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2 \rangle .$$

The point A here is initial point and B is terminal one.

A vector with the initial point located at the origin is called the **position** vector (or we say that a vector is in standard position). Vectors are equal if they have the same length and direction (same slope).

EXAMPLE 3. *Graph the vector with initial point A(1, -2) and terminal point B(2,1). Find the components of $\overrightarrow{AB}$ and $\overrightarrow{BA}$.*

Vector operations

- *Scalar Multiplication*: If c is a scalar and $\mathbf{a} = \langle a_1, a_2 \rangle$, then

$$c\mathbf{a} = c\langle a_1, a_2 \rangle = \langle ca_1, ca_2 \rangle .$$

DEFINITION 4. *Two vectors* **a** *and* **b** *are called* **parallel** *if* $\mathbf{b} = c\mathbf{a}$ *with some scalar* c.

If $c > 0$ then a and ca have the same direction, if $c < 0$ then a and ca have the opposite direction.

EXAMPLE 5. *Show that the vectors* $\mathbf{a} = \left\langle \pi x - 1, y^2 + 2y + 1 \right\rangle$ *and* $\mathbf{b} = \left\langle 4\pi x - 4, 4(y+1)^2 \right\rangle$ *are parallel.*

- *Vector addition*: If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle .$$

Triangle Law **Parallelogram Law**

$\mathbf{a} + \mathbf{b}$ is called the **resultant vector**

EXAMPLE 6. *Let* $\mathbf{a} = \langle -1, 2 \rangle$ *and* $\mathbf{b} = \langle 2.1, -0.5 \rangle$. *Then* $3\mathbf{a} + 2\mathbf{b} =$

EXAMPLE 7. *Given the following five points on the plane:*

$$P_1(1,1), P_2(3,7), P_3(2017,-23), P_4(711.7, 2333.5), P_5(2,4).$$

Compute
$2\overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \overrightarrow{P_3P_4} + \overrightarrow{P_4P_5} + 3\overrightarrow{P_5P_1}$

Proofs of geometric facts via vector techniques.

EXAMPLE 8. *A quadrilateral has one pair of opposite sides parallel and of equal length. Use vectors to prove that the other pair of opposite sides is parallel and of equal length.*

EXAMPLE 9. *Use vectors to prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long.*

EXAMPLE 10. *Use vectors to prove that the midpoints of the sides of a quadrilateral are the vertices of a parallelogram.*

Norm of a vector

The **magnitude**, the **norm**, or **length** of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is denoted by $|\mathbf{a}|$,

$$|\mathbf{a}| =$$

EXAMPLE 11. *Find:* $|\langle 3, -8 \rangle|$, $\left| \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle \right|$, $|\mathbf{0}|$

Unit vectors

A **unit** vector is a vector with length one. The **standard basis vectors** are given by the unit vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ along the x and y directions, respectively. Using the basis vectors, one can represent any vector $\mathbf{a} = \langle a_1, a_2 \rangle$ as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$

EXAMPLE 12. *Given* $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \langle 5, -2\rangle$. *Find a scalars* s *and* t *such that* $s\mathbf{a} + t\mathbf{b} = -4\mathbf{j}$.

Normalizing a vector

Any vector can be made into a unit vector by dividing it by its length. So, a unit vector in the direction of $\mathbf{a}$ is

$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

The process of multiplying a vector $\mathbf{a}$ by the reciprocal of its length to obtain a unit vector with the same direction is called **normalizing a**.

Any vector $\mathbf{a}$ can be fully represented by providing its length, $|\mathbf{a}|$ and a unit vector $\widehat{\mathbf{a}}$ in its direction:

$$\mathbf{a} = |\mathbf{a}|\,\widehat{\mathbf{a}},$$

i.e. *any vector is equal to its length times a unit vector in the same direction.*

EXAMPLE 13. *Given* $\mathbf{a} = \langle 2, -1\rangle$. *Find*

(a) *a unit vector that has the same direction as* $\mathbf{a}$*;*

(b) *a vector* $\mathbf{b}$ *in the direction opposite to* $\mathbf{a}$ *s.t* $|\mathbf{b}| = 7$.

Vectors determined by length and angle

The direction of a vector is the positive angle θ formed by the positive x-axis and the vector.

If $\mathbf{a}$ is a nonzero position vector on the xy-plane that makes a positive angle θ with the positive x-axis then $\mathbf{a}$ can be expressed in trigonometric form as

$$\mathbf{a} = |\mathbf{a}|\,\langle \cos\theta, \sin\theta\rangle\,.$$

For a unit vector this simplifies to

$$\widehat{\mathbf{a}} = \langle \cos\theta, \sin\theta \rangle$$

EXAMPLE 14. *Write in component form the vector of length 5 and with direction* $3\pi/4$.

EXAMPLE 15. *Find the direction of the vector* $\mathbf{a} = -\mathbf{i} + \mathbf{j}$.

Quantities such as force, displacement or velocity that have direction as well as magnitude are represented by vectors.

The velocity of an object can be modeled by a vector, where the direction of the vector is the direction of motion, and the magnitude of the vector is the speed.

EXAMPLE 16. *Ben walks due west on the deck of a ship at 5 mph. The ship is moving north at a speed of 25 mph. Find the direction and speed of Ben relative to the surface of the water.*

EXAMPLE 17. *Two forces F_1 and F_2 with magnitudes 14 pounds and 12 pounds act on an object at a point P as shown. Find the resultant force as well as it's magnitude and direction.*