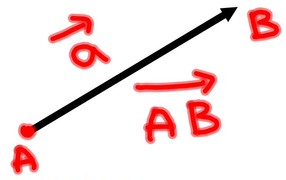
Section 1.1: Vectors

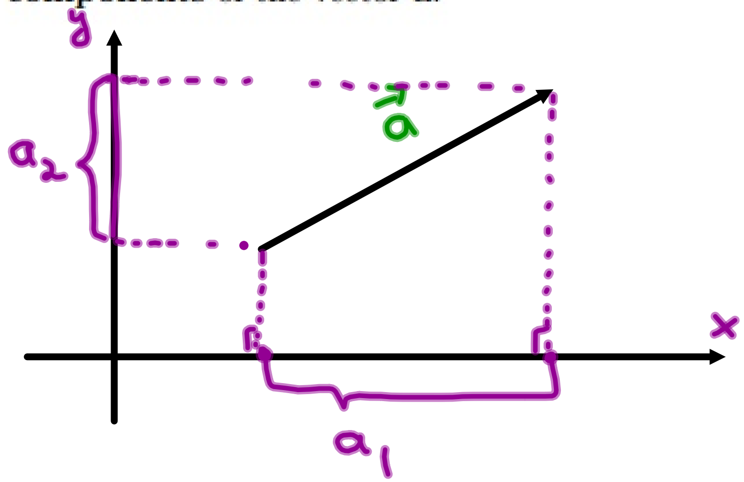
Quantities that we measure that have magnitude but not direction are called **scalars**.



DEFINITION 1. *A **vector** is a quantity that has both* length *magnitude and direction.*

Vectors are drawn as directed line segments and typically are denoted by a boldfaced character or a character with and arrow on it (i.e. **a** or $\overrightarrow{a}$).

A *2-dimensional vector* is an ordered pair $\mathbf{a} = \langle a_1, a_2 \rangle$. The numbers a_1 and a_2 are called the **components** of the vector **a**.



DEFINITION 2. *Given the points $A(a_1, a_2)$ and $B(b_1, b_2)$, the vector* **a** *with representation $\overrightarrow{AB}$ is*

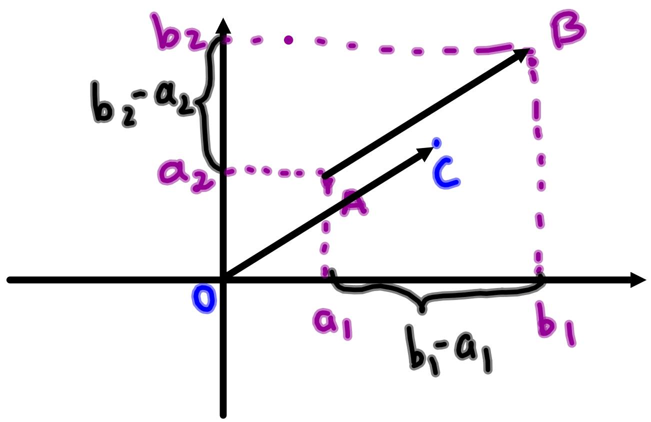
$$\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2 \rangle .$$

(informally, B − A)

(formally, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

The point A here is initial point and B is terminal one.

A vector with the initial point located at the origin is called the **position** vector (or we say that a vector is in standard position). Vectors are equal if they have the same length and direction (same slope).



$$\underbrace{\overrightarrow{OC}}_{\text{the position vector}} = \overrightarrow{AB}$$

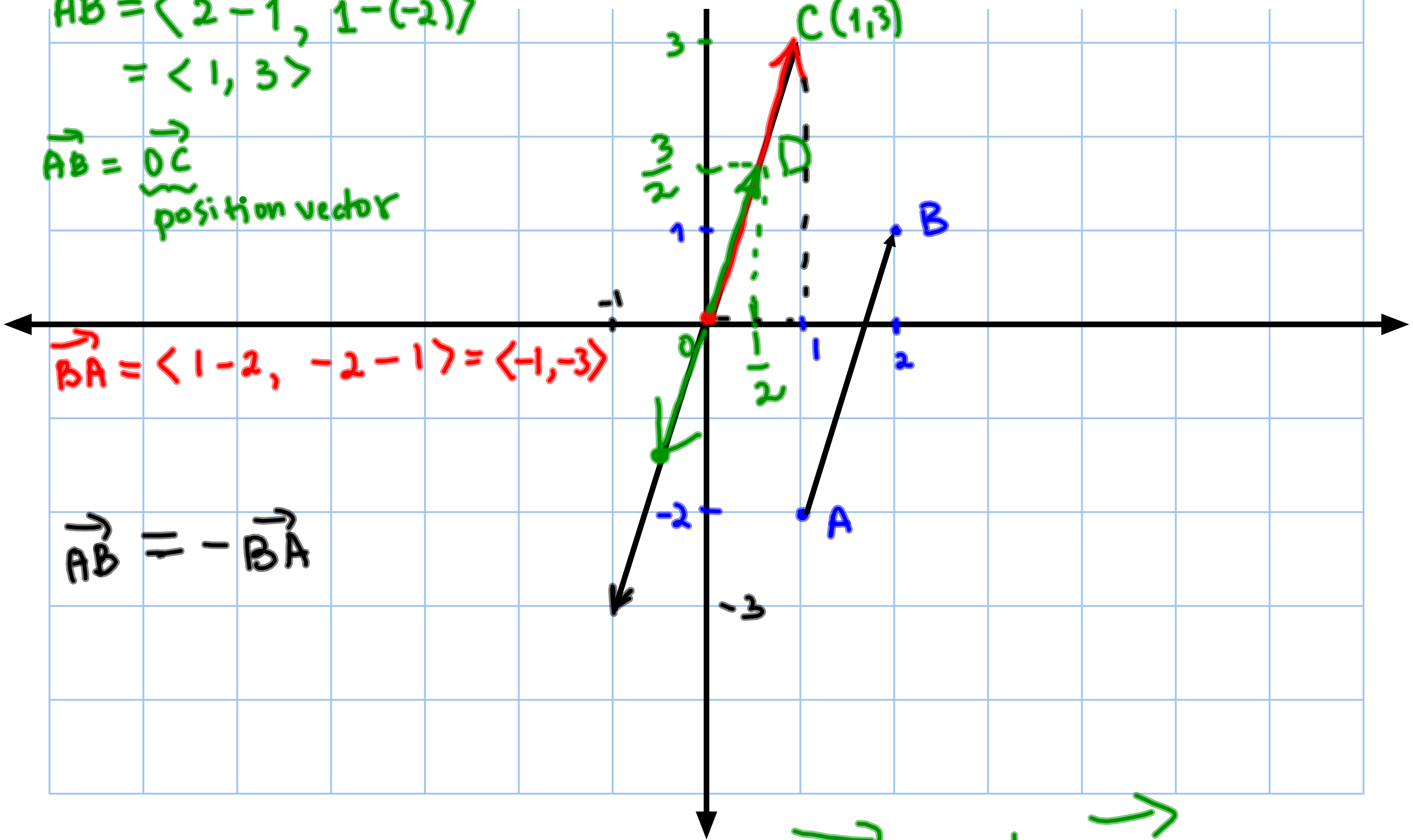
EXAMPLE 3. *Graph the vector with initial point A(1, -2) and terminal point B(2,1). Find the components of* $\overrightarrow{AB}$ *and* $\overrightarrow{BA}$.

$\overrightarrow{AB} = \langle 2-1,\ 1-(-2)\rangle$
$= \langle 1, 3\rangle$

$\overrightarrow{AB} = \underbrace{\overrightarrow{OC}}_{\text{position vector}}$

$\overrightarrow{BA} = \langle 1-2,\ -2-1\rangle = \langle -1, -3\rangle$

$\overrightarrow{AB} = -\overrightarrow{BA}$



$\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OC}$

Vector operations

$\frac{1}{2}\langle 1,3\rangle = \langle \frac{1}{2}, \frac{3}{2}\rangle$

$-\frac{1}{2}\langle 1,3\rangle = \langle -\frac{1}{2}, -\frac{3}{2}\rangle$

- *Scalar Multiplication*: If c is a scalar and $\mathbf{a} = \langle a_1, a_2\rangle$, then

$$c\mathbf{a} = c\,\langle a_1, a_2\rangle = \langle ca_1, ca_2\rangle .$$

DEFINITION 4. *Two vectors* **a** *and* **b** *are called* **parallel** *if* $\mathbf{b} = c\mathbf{a}$ *with some scalar* c.

If $c > 0$ then $\vec{a}$ and $c\vec{a}$ have the same direction, if $c < 0$ then $\vec{a}$ and $c\vec{a}$ have the opposite direction.

$\langle -2,-4\rangle \parallel \langle 4, 8\rangle$, because $\langle 4,8\rangle = -2\langle -2,-4\rangle$

$\langle 2,-4\rangle \nparallel \langle 4,8\rangle$ If they parallel, then $\langle 2,-4\rangle = c\langle 4,8\rangle$ for some c. Then $2 = 4c \Rightarrow c = \frac{1}{2}$
$-4 = 8c \Rightarrow c = -\frac{1}{2}$ $\Rightarrow \frac{1}{2} = -\frac{1}{2}$, a contradiction

$\langle 5, 0\rangle \parallel \langle 3,0\rangle$

$\langle 5,0\rangle \nparallel \langle 3, 3\rangle$

EXAMPLE 5. *Show that the vectors* $\mathbf{a} = \langle \pi x - 1, y^2 + 2y + 1\rangle$ *and* $\mathbf{b} = \langle 4\pi x - 4, 4(y+1)^2\rangle$ *are parallel.*

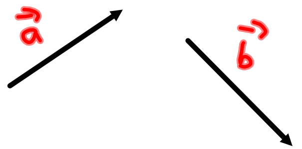
Way 1 $\vec{b} = \langle 4\pi x - 4, 4(y+1)^2\rangle = \langle 4(\pi x - 1), 4(y^2+2y+1)\rangle$
$= 4\langle \pi x - 1, y^2+2y+1\rangle = 4\vec{a}$

Way 2 $\frac{1}{4} = \frac{\pi x - 1}{4\pi x - 4} = \frac{y^2+2y+1}{4(y+1)^2} = \frac{(y+1)^2}{4(y+1)^2} = \frac{1}{4}$

- *Vector addition*: If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ then

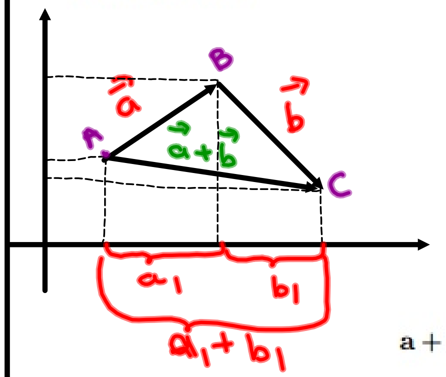
$$\mathbf{a} + \mathbf{b} = \langle \underbrace{a_1 + b_1}, a_2 + b_2 \rangle .$$

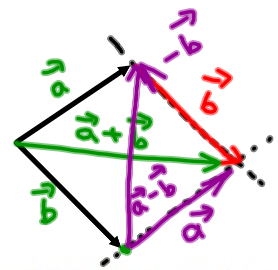
$\vec{AB} + \vec{BC} = \vec{AC}$

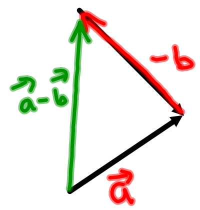


$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

Triangle Law **Parallelogram Law**



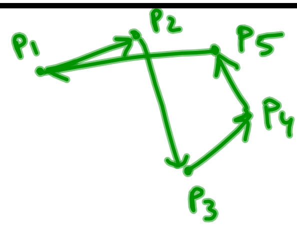




$\mathbf{a} + \mathbf{b}$ is called the **resultant vector**

EXAMPLE 6. *Let* $\mathbf{a} = \langle -1, 2 \rangle$ *and* $\mathbf{b} = \langle 2.1, -0.5 \rangle$. *Then* $3\mathbf{a} + 2\mathbf{b} =$

EXAMPLE 7. *Given the following five points on the plane:*



$$P_1(1,1), P_2(3,7), P_3(2017,-23), P_4(711.7, 2333.5), P_5(2,4).$$

Compute

$$2\overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \overrightarrow{P_3P_4} + \overrightarrow{P_4P_5} + 3\overrightarrow{P_5P_1} =$$

$$\overrightarrow{P_1P_2} + \overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \overrightarrow{P_3P_4} + \overrightarrow{P_4P_5} + \overrightarrow{P_5P_1} + \overrightarrow{P_5P_1} + \overrightarrow{P_5P_1} =$$

$$\overrightarrow{P_1P_1} = \vec{0}$$

$$\overrightarrow{P_5P_1} + \overrightarrow{P_1P_2} = \overrightarrow{P_5P_2}$$

$$\overrightarrow{P_5P_2} + \overrightarrow{P_5P_1}$$

$$= \langle 3-2, 7-4\rangle + \langle 1-2, 1-4\rangle$$

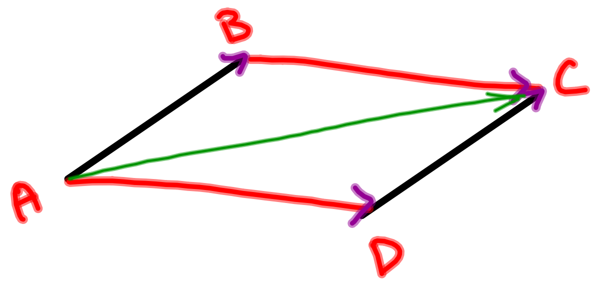
$$= \langle 1, 3\rangle + \langle -1, -3\rangle$$

$$= \langle 1+(-1), 3+(-3)\rangle$$

$$= \langle 0, 0\rangle = \vec{0}$$

Proofs of geometric facts via vector techniques.

EXAMPLE 8. *A quadrilateral has one pair of opposite sides parallel and of equal length. Use vectors to prove that the other pair of opposite sides is parallel and of equal length.* (in other words, prove that this quadrilateral is a parallelogram).

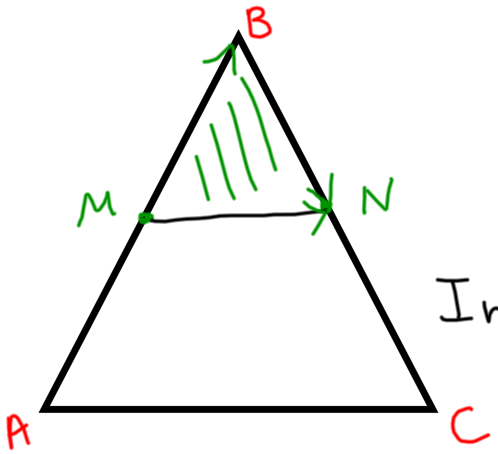


Given: $\overrightarrow{AB} = \overrightarrow{DC} \Rightarrow \boxed{\overrightarrow{CD} = \overrightarrow{BA} \quad (1)}$

Show: $\overrightarrow{AD} = \overrightarrow{BC}$

Proof $\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AC} + \overrightarrow{BA} = \overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BC}$.

EXAMPLE 9. *Use vectors to prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long.*



Given: $\overline{AM} = \overline{MB}$ and $\overline{BN} = \overline{NC}$

Prove: $\overline{MN} = \frac{1}{2}\overline{AC}$ and $MN \parallel AC$

In other words

Given: $\overrightarrow{AM} = \overrightarrow{MB}$ and $\overrightarrow{BN} = \overrightarrow{NC}$

OR

Given: $\overrightarrow{MB} = \frac{1}{2}\overrightarrow{AB}$

(*) $\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BC}$

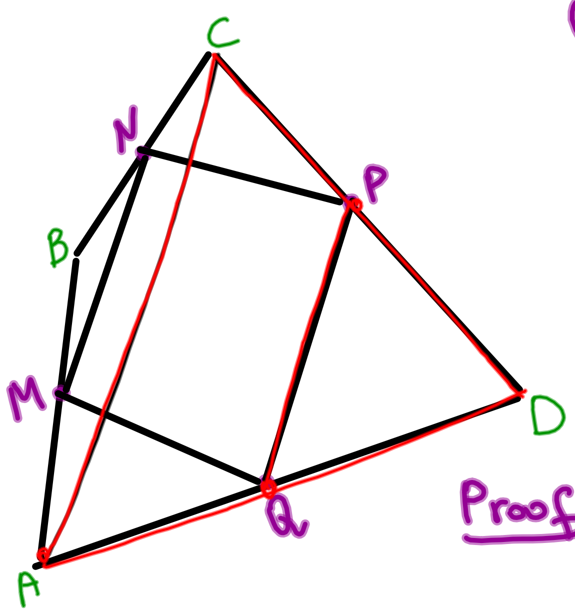
Prove: $\underbrace{\overrightarrow{MN}} = \frac{1}{2}\overrightarrow{AC}$

Proof

$$\overrightarrow{MN} \overset{\Delta\text{ law}}{=} \overrightarrow{MB} + \overrightarrow{BN} \overset{(*)}{=} \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$$

$$= \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) \overset{\Delta\text{ Law}}{=} \frac{1}{2}\overrightarrow{AC}.$$

EXAMPLE 10. *Use vectors to prove that the midpoints of the sides of a quadrilateral are the vertices of a parallelogram.*



Given: M, N, P, Q are midpoints of AB, BC, CD, DA respectively.

Prove: $\overline{MN} = \overline{PQ}$ and $MN \parallel PQ$ and $\overline{MQ} = \overline{NP}$ and $MQ \parallel NP$

In other words, prove $\overrightarrow{MN} = \overrightarrow{PQ}$ and $\overrightarrow{MQ} = \overrightarrow{NP}$

Proof

$$\left.\begin{array}{l} \overrightarrow{MN} \overset{Ex.9}{=} \frac{1}{2}\overrightarrow{AC} \quad (\text{considering } \triangle ABC) \\ \overrightarrow{PQ} \overset{Ex.9}{=} \frac{1}{2}\overrightarrow{AC} \quad (\text{consider } \triangle ADC) \end{array}\right\} \Rightarrow$$

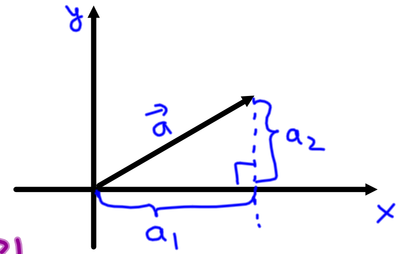
$$\Rightarrow \overrightarrow{MN} = \overrightarrow{PQ}.$$

$\overrightarrow{MQ} = \overrightarrow{NP}$ follows from Ex. 10 (or apply the above approach to $\triangle ABD$ and $\triangle BCD$).

Norm of a vector

The **magnitude**, the **norm**, or **length** of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is denoted by $|\mathbf{a}|$,

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$



$$|c\vec{a}| = |c\langle a_1, a_2\rangle| = |\langle ca_1, ca_2\rangle|$$

$$= \sqrt{(ca_1)^2 + (ca_2)^2} = \sqrt{c^2a_1^2 + c^2a_2^2}$$

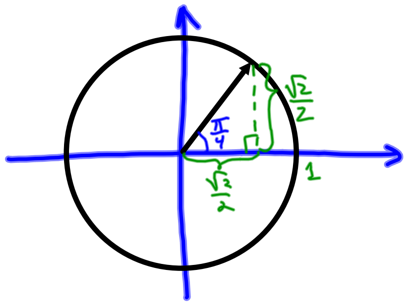
$$= \sqrt{c^2(a_1^2 + a_2^2)} = \sqrt{c^2}\,\sqrt{a_1^2 + a_2^2} = |c| \cdot |\vec{a}|$$

$$\boxed{|c\vec{a}| = |c| \cdot |\vec{a}|}$$

EXAMPLE 11. *Find:* $|\langle 3, -8\rangle|$, $\left|\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle\right|$, $|\mathbf{0}|$

$$|\langle 3, -8\rangle| = \sqrt{3^2 + (-8)^2} = \sqrt{9 + 64} = \sqrt{73}$$

$$\left|\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle\right| = 1$$



$$|\vec{0}| = |\langle 0, 0\rangle| = 0$$

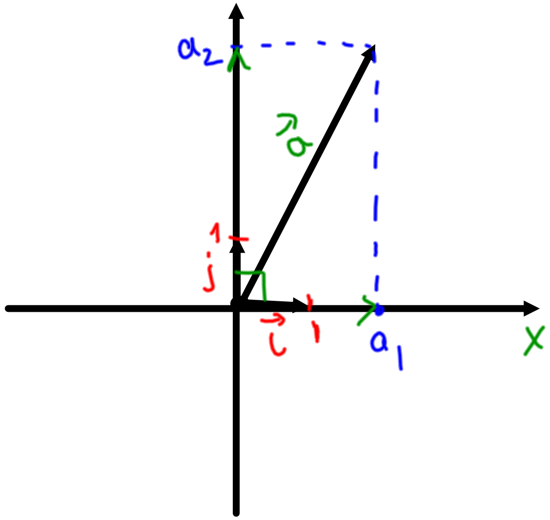
$$\left|\left\langle -\frac{2500}{3}, -\frac{100}{3}\right\rangle\right| = \left|-\frac{100}{3}\langle 25, 1\rangle\right| = \frac{100}{3}\sqrt{25^2 + 1^2}$$

$$= \frac{100}{3}\sqrt{626}$$

Unit vectors

A **unit** vector is a vector with length one. The **standard basis vectors** are given by the unit vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ along the x and y directions, respectively. Using the basis vectors, one can represent any vector $\mathbf{a} = \langle a_1, a_2 \rangle$ as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} = \langle a_1, a_2 \rangle$$



$$\vec{a} = \langle a_1, a_2 \rangle = \langle \overset{= a_1 \cdot 1}{a_1}, \overset{= a_1 \cdot 0}{0} \rangle + \langle 0, a_2 \rangle$$

$$= a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle$$

$$= a_1 \vec{i} + a_2 \vec{j}$$

$$\langle 3, -\tfrac{1}{2} \rangle = 3\vec{i} - \tfrac{1}{2}\vec{j}$$

$$\langle 0, 15 \rangle = 15\vec{j}$$

EXAMPLE 12. *Given* $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \langle 5, -2\rangle$. *Find a scalars* s *and* t *such that* $\boxed{s\mathbf{a} + t\mathbf{b} = -4\mathbf{j}}$.

$\vec{a} = \langle 2, -1 \rangle$

$\vec{j} = \langle 0, 1 \rangle$

$$s\vec{a} + t\vec{b} = -4\vec{j}$$

$$s\langle 2, -1\rangle + t\langle 5, -2\rangle = -4\langle 0, 1\rangle$$

$$\langle 2s, -s\rangle + \langle 5t, -2t\rangle = \langle 0, -4\rangle$$

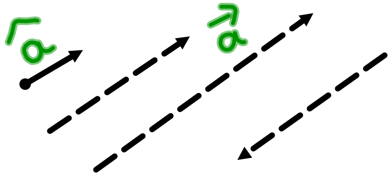
$$\langle \underbrace{2s + 5t}, \overbrace{-s-2t} \rangle = \langle \underbrace{0}, \overbrace{-4} \rangle$$

$$\begin{cases} 2s + 5t = 0 \\ -s - 2t = -4 \quad (\times 2) \end{cases} \Rightarrow + \begin{array}{r} 2s + 5t = 0 \\ -2s - 4t = -8 \\ \hline \boxed{t = -8} \end{array}$$

$$-s - 2(-8) = -4$$

$$-s + 16 = -4$$

$$-s = -20 \Rightarrow \boxed{s = 20}$$



$$|\hat{a}| = \left| \frac{1}{|\vec{a}|} \cdot \vec{a} \right| = \frac{1}{|\vec{a}|} \cdot |\vec{a}| = 1.$$

Normalizing a vector

Any vector can be made into a unit vector by dividing it by its length. So, a unit vector in the direction of **a** is

$$\boxed{\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}} = \frac{1}{|\vec{a}|} \cdot \vec{a}$$

The process of multiplying a vector **a** by the reciprocal of its length to obtain a unit vector with the same direction is called **normalizing a**.

Any vector **a** can be fully represented by providing its length, $|\mathbf{a}|$ and a unit vector $\hat{\mathbf{a}}$ in its direction:

$$\boxed{\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{a}},}$$

i.e. *any vector is equal to its length times a unit vector in the same direction.*

EXAMPLE 13. *Given* $\mathbf{a} = \langle 2, -1 \rangle$. *Find*

(a) *a unit vector that has the same direction as* **a**; (equivalently, normalize $\vec{a}$)

$$|\vec{a}| = |\langle 2, -1 \rangle| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\langle 2, -1 \rangle}{\sqrt{5}} = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

OR

$$= \frac{\sqrt{5}\langle 2, -1 \rangle}{\sqrt{5}\cdot\sqrt{5}} = \frac{\sqrt{5}\langle 2, -1 \rangle}{5} = \left\langle \frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5} \right\rangle$$

(b) *a vector* **b** *in the direction opposite to* **a** *s.t* $|\mathbf{b}| = 7$.

$$\vec{b} = |\vec{b}| \cdot \hat{b} = 7\hat{b} = 7 \cdot (-\hat{a}) = -7\hat{a} =$$

$$= -7\left\langle \frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5} \right\rangle = \boxed{\left\langle \frac{-14\sqrt{5}}{5}, \frac{7\sqrt{5}}{5} \right\rangle}$$

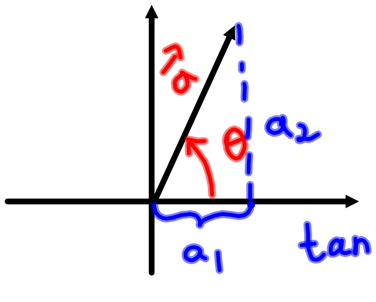
Vectors determined by length and angle

If **a** is a nonzero position vector on the xy-plane that makes an angle θ with the positive x-axis then **a** can be expressed in trigonometric form as $\vec{a} = |\vec{a}| \cdot \hat{a}$

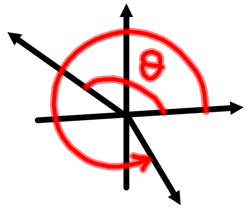
direction of $\vec{a}$

$$\boxed{\mathbf{a} = |\mathbf{a}| \langle \cos\theta, \sin\theta \rangle = \langle |a| \cos\theta, \ |a| \sin\theta \rangle}$$

$$|\langle \cos\theta, \sin\theta \rangle| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$



$$\tan\theta = \frac{a_2}{a_1}$$



For a unit vector this simplifies to

$$\hat{\mathbf{a}} = \langle \cos\theta, \sin\theta \rangle$$

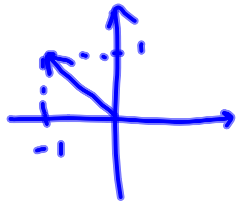
EXAMPLE 14. *Write in component form the vector of length 5 and with direction $3\pi/4$.*

$$\vec{a} = |\vec{a}| \langle \cos\theta, \sin\theta \rangle = 5 \langle \cos\frac{3\pi}{4}, \sin\frac{3\pi}{4} \rangle$$

$$= 5 \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \langle -\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2} \rangle$$

EXAMPLE 15. *Find the direction of the vector* $\mathbf{a} = -\mathbf{i} + \mathbf{j}$.

$$\vec{a} = -i + j = \langle -1, 1 \rangle$$



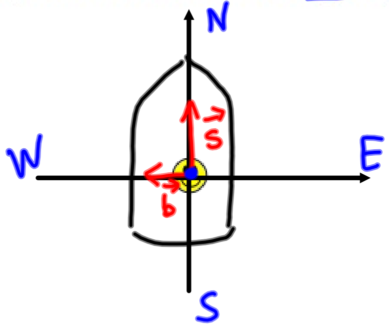
$$\tan\theta = \frac{1}{-1} = -1, \quad \frac{\pi}{2} < \theta < \pi$$

$$\boxed{\theta = \frac{3\pi}{4}}$$

Quantities such as force, displacement or velocity that have direction as well as magnitude are represented by vectors.

The velocity of an object can be modeled by a vector, where the direction of the vector is the direction of motion, and the magnitude of the vector is the speed. $= |\vec{v}|$ mi/h

EXAMPLE 16. *Ben walks due west on the deck of a ship at 5 mph. The ship is moving north at a speed of 25 mph. Find the direction and speed of Ben relative to the surface of the water.*



$\vec{b}$ is velocity of Ben

$|\vec{b}| = 5$ mi/h

$\vec{b} = 5\langle -1,0\rangle = \langle -5,0\rangle$

$\vec{s}$ is velocity of the ship

$|\vec{s}| = 25$ mi/h.

$\vec{s} = 25\cdot\langle 0,1\rangle = \langle 0,25\rangle$

the resultant velocity $\vec{r} = \vec{b} + \vec{s} = \langle -5,0\rangle + \langle 0,25\rangle = \langle -5,25\rangle = 5\langle -1,5\rangle$

speed $= |\vec{r}| = |5\langle -1,5\rangle| = 5|\langle -1,5\rangle|$

$= 5\sqrt{(-1)^2+5^2} = \boxed{5\sqrt{26} \text{ mi/h}}$

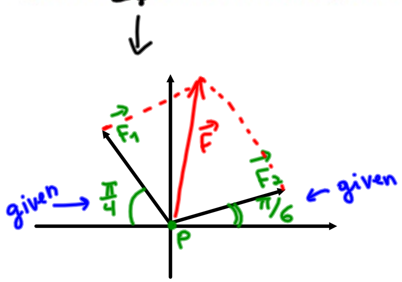
θ is direction of r:

$\tan\theta = \frac{25}{-5} = \frac{5}{-1} = -5$, where $\frac{\pi}{2} < \theta < \pi$

$\theta = \arctan(-5) = -78.69° + 180°$

$= \boxed{101.31°}$

EXAMPLE 17. *Two forces F_1 and F_2 with magnitudes 14 pounds and 12 pounds act on an object at a point P as shown. Find the resultant force as well as it's magnitude and direction.*



$|\vec{F_1}| = 14$ lb , $|\vec{F_2}| = 12$ lb

$\theta_1 = \frac{3\pi}{4}$ $\quad \theta_2 = \frac{\pi}{6}$

$\vec{F_1} = |\vec{F_1}| \langle \cos\theta_1, \sin\theta_1 \rangle$

$= 14 \langle \cos\frac{3\pi}{4}, \sin\frac{3\pi}{4} \rangle$

$= 14 \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$\vec{F_1} = \langle -7\sqrt{2}, 7\sqrt{2} \rangle$

$\vec{F_2} = |\vec{F_2}| \langle \cos\theta_2, \sin\theta_2 \rangle = 12 \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \rangle$

$= 12 \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = \langle 6\sqrt{3}, 6 \rangle$

the resultant force

$\vec{F} = \vec{F_1} + \vec{F_2} = \langle -7\sqrt{2}, 7\sqrt{2} \rangle + \langle 6\sqrt{3}, 6 \rangle$

$= \langle -7\sqrt{2} + 6\sqrt{3}, 7\sqrt{2} + 6 \rangle$

$= \langle 6\sqrt{3} - 7\sqrt{2}, 7\sqrt{2} + 6 \rangle$

$|\vec{F}| = \sqrt{(6\sqrt{3} - 7\sqrt{2})^2 + (7\sqrt{2} + 6)^2} = \dots$ lb

θ is direction of $\vec{F}$:

$\tan\theta = \frac{7\sqrt{2} + 6}{6\sqrt{3} - 7\sqrt{2}} \Rightarrow \theta = \dots°$