Section 1.2: The Dot Product

DEFINITION 1. *The* **dot product** *of two given vectors* $\mathbf{a} = \langle a_1, a_2 \rangle$ *and* $\mathbf{b} = \langle b_1, b_2 \rangle$ *is the number*

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

EXAMPLE 2. *Compute the dot product of* $\mathbf{a} = \langle 2, -3 \rangle$ *and* $\mathbf{b} = \langle 3, -4 \rangle$.

Algebraic properties

THEOREM 3. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and α ia a scalar, then

(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(c) $\alpha(\mathbf{a} \cdot \mathbf{b}) = (\alpha \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha \mathbf{b})$

(d) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

(e) $\mathbf{0} \cdot \mathbf{a} = 0$

Proof.

The property (d) of Theorem 3 implies a useful way of expressing the length of a vector in terms of dot product:

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

THEOREM 4. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are two nonzero vectores and if* θ *is the angle between them, then*

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \tag{1}$$

Note that the proof of the above theorem can be obtain by using the law of cosines and the algebraic properties of dot product.

EXAMPLE 5. *Determine the angle between* $\mathbf{a} = \langle 2, -3 \rangle$ *and* $\mathbf{b} = \langle 3, -4 \rangle$.

It will often be convenient to express (1) as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta, \tag{2}$$

where $0 \leq \theta \leq \pi$.

The dot product gives us a simple way for determining if two vectors are perpendicular (or orthogonal), namely,

THEOREM 6. *Two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal if and only if* $\mathbf{a} \cdot \mathbf{b} = 0$.

Proof.

EXAMPLE 7. *What is the dot product of* $12\mathbf{j}$ *and* $11\mathbf{i}$*?*

EXAMPLE 8. *Determine whether the given vectors are orthogonal, parallel, or neither. If the vectors are non orthogonal and non parallel, then determine whether the angle between them is acute or obtuse.*

(a) $\langle 3, 4 \rangle$, $\langle -8, 6 \rangle$

(b) $\langle -7, -4\rangle$, $\langle 28, 16\rangle$

(c) $\langle -1, 1\rangle$, $\langle 2, -3\rangle$

DEFINITION 9. *The* **work** *done by a force* $\mathbf{F}$ *in moving and object from point* A *to point* B *is given by*

$$W = \mathbf{F} \cdot \mathbf{D}$$

where $\mathbf{D} = \overrightarrow{AB}$ *is the distance the object has moved (or displacement).*

Question: If you push against a wall, you may tire yourself out, but you will not perform any work. Why?

EXAMPLE 10. *A wagon is pulled horizontally by exerting a force of* $50lb$ *on the handle at an angle* 30^o *with the horizontal. How much work is done in moving the wagon* $10ft$.

EXAMPLE 11. *A constant force* $\mathbf{F} = 25\mathbf{i} + 4\mathbf{j}$ *(the magnitude of* $\mathbf{F}$ *is measured in Newtons) is used to move an object from* $A(1, 1)$ *to* $B(5, 6)$. *Find the work done if the distance is measured in meters.*

DEFINITION 12. *The* **orthogonal compliment of** $\mathbf{a} = \langle a_1, a_2 \rangle$ **is** $\mathbf{a}^{\perp} = \langle -a_2, a_1 \rangle$.

Note that $|\mathbf{a}| = \left|\mathbf{a}^{\perp}\right|$ and $\mathbf{a} \cdot \mathbf{a}^{\perp} =$

EXAMPLE 13. *Given* $\langle 4, -2 \rangle$, $\langle 2, -1 \rangle$, $\langle -2, 1 \rangle$ *and* $\mathbf{a} = \langle 1, 2 \rangle$. *Which of these vectors is*

- *orthogonal to* **a***?*

- *the orthogonal compliment of* **a***?*

Scalar and vector projections: For given two vectors **a** and **b** we determine the projection of **b** onto **a**.

- The vector projection of **b** onto **a** is denotes by $\text{proj}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}.$$

- The scalar projection of **b** onto **a** (or the component of **b** along **a**) is denotes by $\text{comp}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}.$$

EXAMPLE 14. *Given* $\mathbf{a} = \langle 4, 3\rangle$ *and* $\mathbf{b} = \langle 1, -1\rangle$. *Find:*

- $\mathbf{a} \cdot \mathbf{b} =$
- $|\mathbf{a}| =$
- $|\mathbf{b}| =$
- $\text{proj}_{\mathbf{b}}\mathbf{a} =$
- $\text{comp}_{\mathbf{a}}\mathbf{b} =$

EXAMPLE 15. *Find the distance from the point* $P(-2, 3)$ *to the line* $y = 3x + 5$.