

## Section 1.2: The Dot Product

DEFINITION 1. The **dot product** of two given vectors  $\mathbf{a} = \langle a_1, a_2 \rangle$  and  $\mathbf{b} = \langle b_1, b_2 \rangle$  is the number

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

EXAMPLE 2. Compute the dot product of  $\mathbf{a} = \langle 2, -3 \rangle$  and  $\mathbf{b} = \langle 3, -4 \rangle$ .

### Algebraic properties

THEOREM 3. If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors and  $\alpha$  is a scalar, then

(a)  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(b)  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(c)  $\alpha(\mathbf{a} \cdot \mathbf{b}) = (\alpha\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha\mathbf{b})$

(d)  $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

(e)  $\mathbf{0} \cdot \mathbf{a} = 0$

*Proof.*

The property (d) of Theorem 3 implies a useful way of expressing the length of a vector in terms of dot product:

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

THEOREM 4. If  $\mathbf{a}$  and  $\mathbf{b}$  are two nonzero vectors and if  $\theta$  is the angle between them, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \quad (1)$$

Note that the proof of the above theorem can be obtained by using the law of cosines and the algebraic properties of dot product.

EXAMPLE 5. Determine the angle between  $\mathbf{a} = \langle 2, -3 \rangle$  and  $\mathbf{b} = \langle 3, -4 \rangle$ .

It will often be convenient to express (1) as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta, \quad (2)$$

where  $0 \leq \theta \leq \pi$ .

The dot product gives us a simple way for determining if two vectors are perpendicular (or orthogonal), namely,

THEOREM 6. Two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal if and only if  $\mathbf{a} \cdot \mathbf{b} = 0$ .

*Proof.*

EXAMPLE 7. What is the dot product of  $12\mathbf{j}$  and  $11\mathbf{i}$ ?

EXAMPLE 8. Determine whether the given vectors are orthogonal, parallel, or neither. If the vectors are non orthogonal and non parallel, then determine whether the angle between them is acute or obtuse.

(a)  $\langle 3, 4 \rangle$ ,  $\langle -8, 6 \rangle$

(b)  $\langle -7, -4 \rangle, \langle 28, 16 \rangle$

(c)  $\langle -1, 1 \rangle, \langle 2, -3 \rangle$

DEFINITION 9. The **work** done by a force  $\mathbf{F}$  in moving an object from point  $A$  to point  $B$  is given by

$$W = \mathbf{F} \cdot \mathbf{D}$$

where  $\mathbf{D} = \overrightarrow{AB}$  is the distance the object has moved (or displacement).

Question: If you push against a wall, you may tire yourself out, but you will not perform any work. Why?

EXAMPLE 10. A wagon is pulled horizontally by exerting a force of 50lb on the handle at an angle  $30^\circ$  with the horizontal. How much work is done in moving the wagon 10ft.

EXAMPLE 11. A constant force  $\mathbf{F} = 25\mathbf{i} + 4\mathbf{j}$  (the magnitude of  $\mathbf{F}$  is measured in Newtons) is used to move an object from  $A(1, 1)$  to  $B(5, 6)$ . Find the work done if the distance is measured in meters.