Section 1.2: The Dot Product

DEFINITION 1. *The* **dot product** *of two given vectors* $\mathbf{a} = \langle a_1, a_2 \rangle$ *and* $\mathbf{b} = \langle b_1, b_2 \rangle$ *is the number*

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

component formula

EXAMPLE 2. *Compute the dot product of* $\mathbf{a} = \langle 2, -3 \rangle$ *and* $\mathbf{b} = \langle 3, -4 \rangle$.

$$\vec{a} \cdot \vec{b} = \langle 2, -3 \rangle \cdot \langle 3, -4 \rangle = 2 \cdot 3 + (-3) \cdot (-4)$$
$$= 6 + 12 = 18$$

$\mathbb{R}^2$ $\quad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

$\mathbb{R}^3$ $\quad \vec{a} = \langle a_1, a_2, a_3 \rangle, \quad \vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$\mathbb{R}^n$ $\quad \vec{a} = \langle a_1, a_2, \dots, a_n \rangle, \quad \vec{b} = \langle b_1, b_2, \dots, b_n \rangle$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Algebraic properties

THEOREM 3. If $\mathbf{a} = \langle a_1, a_2 \rangle$, $\mathbf{b} = \langle b_1, b_2 \rangle$, and $\mathbf{c} = \langle c_1, c_2 \rangle$ are vectors and α ia a scalar, then

(a) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ Commutative

(b) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ Distributive

(c) $\boxed{\alpha(\mathbf{a}\cdot\mathbf{b}) = (\alpha\mathbf{a})\cdot\mathbf{b}} = \mathbf{a}\cdot(\alpha\mathbf{b})$

(d) $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$

(e) $\mathbf{0}\cdot\mathbf{a} = 0$

Proof (a) $\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 = b_1 a_1 + b_2 a_2 = \langle b_1, b_2\rangle \cdot \langle a_1, a_2 \rangle = \vec{b}\cdot\vec{a}$

Proof. (c) $\alpha(\vec{a}\cdot\vec{b}) = \alpha(a_1 b_1 + a_2 b_2) = (\alpha a_1) b_1 + (\alpha a_2) b_2 = \langle \alpha a_1, \alpha a_2 \rangle \cdot \langle b_1, b_2 \rangle = (\alpha\vec{a})\cdot\vec{b}$

(b) $\underbrace{\vec{a}\cdot(\vec{b}+\vec{c})} = \vec{a}\cdot(\langle b_1, b_2\rangle + \langle c_1, c_2\rangle)$

$= \langle a_1, a_2\rangle \cdot \langle b_1 + c_1, b_2 + c_2\rangle$

$= a_1(b_1+c_1) + a_2(b_2+c_2)$

$= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2$

$= (a_1 b_1 + a_2 b_2) + (a_1 c_1 + a_2 c_2)$

$= \langle a_1, a_2\rangle\cdot\langle b_1, b_2\rangle + \langle a_1, a_2\rangle \cdot \langle c_1, c_2\rangle$

$= \underbrace{\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}}$

(d) $\vec{a}\cdot\vec{a} = \langle a_1, a_2\rangle\cdot\langle a_1, a_2\rangle = a_1 a_1 + a_2 a_2$

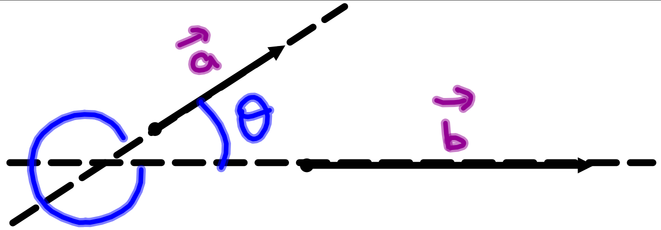
$= a_1^2 + a_2^2 = \left(\sqrt{a_1^2 + a_2^2}\right)^2 = |\vec{a}|^2$

The property (d) of Theorem 3 implies a useful way of expressing the length of a vector in terms of dot product:

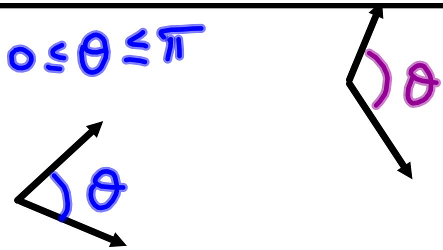
$\vec{a}\cdot\vec{a} = |\vec{a}|^2 \Longrightarrow$ $\boxed{|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}.}$

(e) $\vec{0}\cdot\vec{a} = \langle 0, 0\rangle \cdot \langle a_1, a_2\rangle = 0\cdot a_1 + 0\cdot a_2 = 0.$

$\square$



$0 \le \theta \le \pi$



THEOREM 4. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are two nonzero vectores and if* θ *is the angle between them, then*

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{a_1 b_1 + a_2 b_2}{\sqrt{a_1^2+a_2^2}\cdot\sqrt{b_1^2+b_2^2}} \tag{1}$$

always positive

Note that the proof of the above theorem can be obtain by using the law of cosines and the algebraic properties of dot product.

$\vec{a}\cdot\vec{b} > 0 \Rightarrow \cos\theta > 0 \Rightarrow \theta$ is acute

$\vec{a}\cdot\vec{b} < 0 \Rightarrow \cos\theta < 0 \Rightarrow \theta$ is obtuse

EXAMPLE 5. *Determine the angle between* $\mathbf{a} = \langle 2, -3\rangle$ *and* $\mathbf{b} = \langle 3, -4\rangle$.

$$\cos\theta = \frac{\langle 2,-3\rangle \cdot \langle 3,-4\rangle}{|\langle 2,-3\rangle| \cdot |\langle 3,-4\rangle|} = \frac{2\cdot 3 + (-3)\cdot(-4)}{\sqrt{2^2+(-3)^2}\,\sqrt{3^2+(-4)^2}}$$

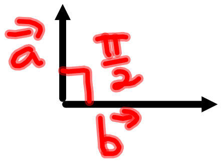
$$= \frac{18}{\sqrt{13}\,\sqrt{25}} = \frac{18}{5\sqrt{13}} = \frac{18\sqrt{13}}{5\cdot 13} = \frac{18\sqrt{13}}{65}$$

$$\theta = \cos^{-1}\left(\frac{18\sqrt{3}}{65}\right)$$

It will often be convenient to express (1) as

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} \implies \boxed{\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta,} \quad \text{geometric formula} \tag{2}$$

where $0 \le \theta \le \pi$.



$\vec{a} \perp \vec{b}$

$\theta = \frac{\pi}{2}$

The dot product gives us a simple way for determining if two vectors are perpendicular (or orthogonal), namely,

$$\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$$

THEOREM 6. *Two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal if and only if* $\mathbf{a} \cdot \mathbf{b} = 0$.

Proof. $(\Rightarrow)$ Given $\vec{a} \perp \vec{b}$. Show $\vec{a} \cdot \vec{b} = 0$

$\Downarrow$

$\theta = \measuredangle \vec{a}, \vec{b} = \frac{\pi}{2}$

$\Downarrow$

$\cos\theta = \cos\frac{\pi}{2} = 0 \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta = |\vec{a}| \cdot |\vec{b}| \cdot 0 = 0$

$(\Leftarrow)$ Given $\vec{a} \cdot \vec{b} = 0$. Show $\vec{a} \perp \vec{b}$

$\Downarrow$

$0 = \vec{a} \cdot \vec{b} = \underbrace{|\vec{a}|}_{\neq 0} \cdot \underbrace{|\vec{b}|}_{\neq 0} \cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

EXAMPLE 7. *What is the dot product of* $12\mathbf{j}$ *and* $11\mathbf{i}$?

$$12\vec{j}\cdot 11\vec{i} \overset{\text{Th.3(c)}}{=} 12\cdot 11\,(\underbrace{\vec{i}\cdot\vec{j}}_{\vec{i}\perp\vec{j}}) = 0$$

$\perp$ $\parallel$

EXAMPLE 8. *Determine whether the given vectors are orthogonal, parallel, or neither. If the vectors are non orthogonal and non parallel, then determine whether the angle between them is acute or obtuse.*

(a) $\langle 3,4\rangle$, $\langle -8,6\rangle$

$$\langle 3,4\rangle\cdot\langle -8,6\rangle = 3\cdot(-8)+4\cdot 6 = 0 \Rightarrow \langle 3,4\rangle \perp \langle -8,6\rangle$$

(b) $\langle -7,-4\rangle$, $\langle 28,16\rangle$

$$\langle 28,16\rangle = \langle -4\cdot(-7), -4\cdot 4\rangle = -4\langle -7,-4\rangle$$

So, $\langle -7,-4\rangle \parallel \langle 28,16\rangle$

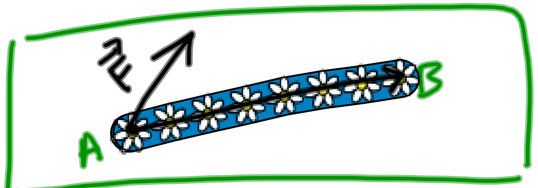
(Also, $-\frac{7}{28} = -\frac{4}{16}$)

(c) $\langle -1,1\rangle$, $\langle 2,-3\rangle$

$$-\frac{1}{2} \neq \frac{1}{-3} \Rightarrow \langle -1,1\rangle \nparallel \langle 2,-3\rangle$$

$\cos\theta < 0$

$$\langle -1,1\rangle\cdot\langle 2,-3\rangle = -1\cdot 2 + 1\cdot(-3) < 0 \Rightarrow \theta \text{ is obtuse}$$



DEFINITION 9. *The* **work** *done by a force* **F** *in moving an~~y~~ object from point A to point B is given by*

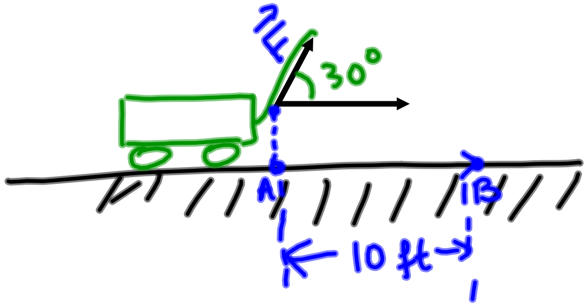
$$W = \mathbf{F}\cdot\mathbf{D} = \vec{F}\cdot\overrightarrow{AB}$$

where $|\mathbf{D}| = |\overrightarrow{AB}|$ *is the distance the object has moved (~~or displacement~~).*

Question: If you push against a wall, you may tire yourself out, but you will not perform any work. Why?

$\vec{D} = \vec{0} \Rightarrow W = \vec{F}\cdot\vec{0} = 0$ (units)

EXAMPLE 10. *A wagon is pulled horizontally by exerting a force of* 50*lb on the handle at an angle* 30^o *with the horizontal. How much work is done in moving the wagon* $10ft$.



Given: $|\vec{F}| = 50$ lb

$\theta = \measuredangle \vec{F}, \overrightarrow{AB} = 30^\circ = \frac{\pi}{6}$

$|\overrightarrow{AB}| = 10$ ft

Find W

Solution:

$$W = \vec{F}\cdot\overrightarrow{AB} = |\vec{F}|\cdot|\overrightarrow{AB}|\cdot\cos\theta$$

$$= 50\cdot 10\cdot\cos\frac{\pi}{6} = 500\,\frac{\sqrt{3}}{2} = 250\sqrt{3}\ \text{ft}\cdot\text{lb}$$

EXAMPLE 11. *A constant force* $\mathbf{F} = 25\mathbf{i} + 4\mathbf{j}$ *(the magnitude of* $\mathbf{F}$ *is measured in Newtons) is used to move an object from* $A(1,1)$ *to* $B(5,6)$*. Find the work done if the distance is measured in meters.*

$$\overrightarrow{AB} = \langle 5-1\ ,\ 6-1 \rangle = \langle 4, 5 \rangle$$

$$W = \vec{F} \cdot \overrightarrow{AB} = \langle 25, 4 \rangle \cdot \langle 4, 5 \rangle$$

$$= 25 \cdot 4 + 4 \cdot 5 = 120 \text{ J}$$