

## Section 2.2: The Limit of a function

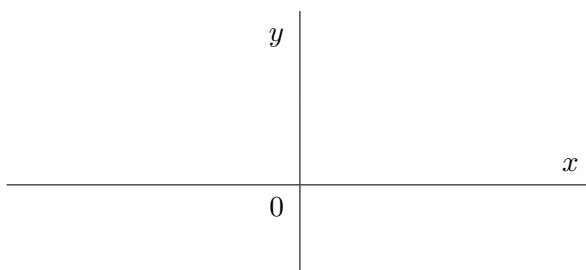
A limit is a way to discuss how the values of a function  $f(x)$  behave when  $x$  approaches a number  $a$ , whether or not  $f(a)$  is defined.

Let's consider the following function:

$$f(x) = \frac{\sin x}{x} \quad (x \text{ in radians}).$$

Note that  $f(0) = \frac{\sin 0}{0}$  is undefined. However, one can compute the values of  $f(x)$  for values of  $x$  close to 0.

$x$	$f(x)$
$\pm 0.1$	0.99833417
$\pm 0.05$	0.99958339
$\pm 0.01$	0.99998333
$\pm 0.005$	0.99999583
$\pm 0.001$	0.99999983



The table allows us to guess (correctly) that that our function gets closer and closer to 1 as  $x$  approaches 0 through positive and negative values. In limit notation it can be written as

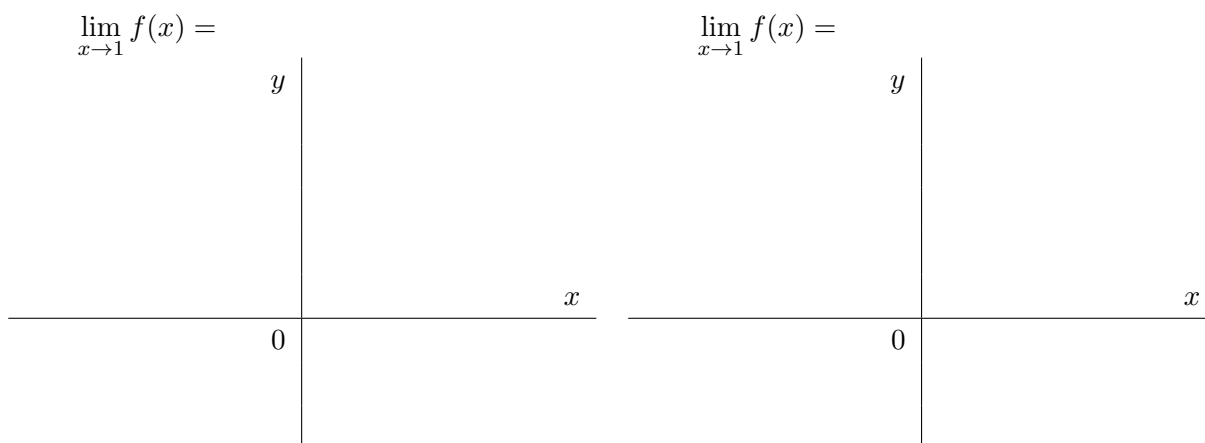
$$\lim_{x \rightarrow 0^-} \frac{\sin x}{x} = \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$

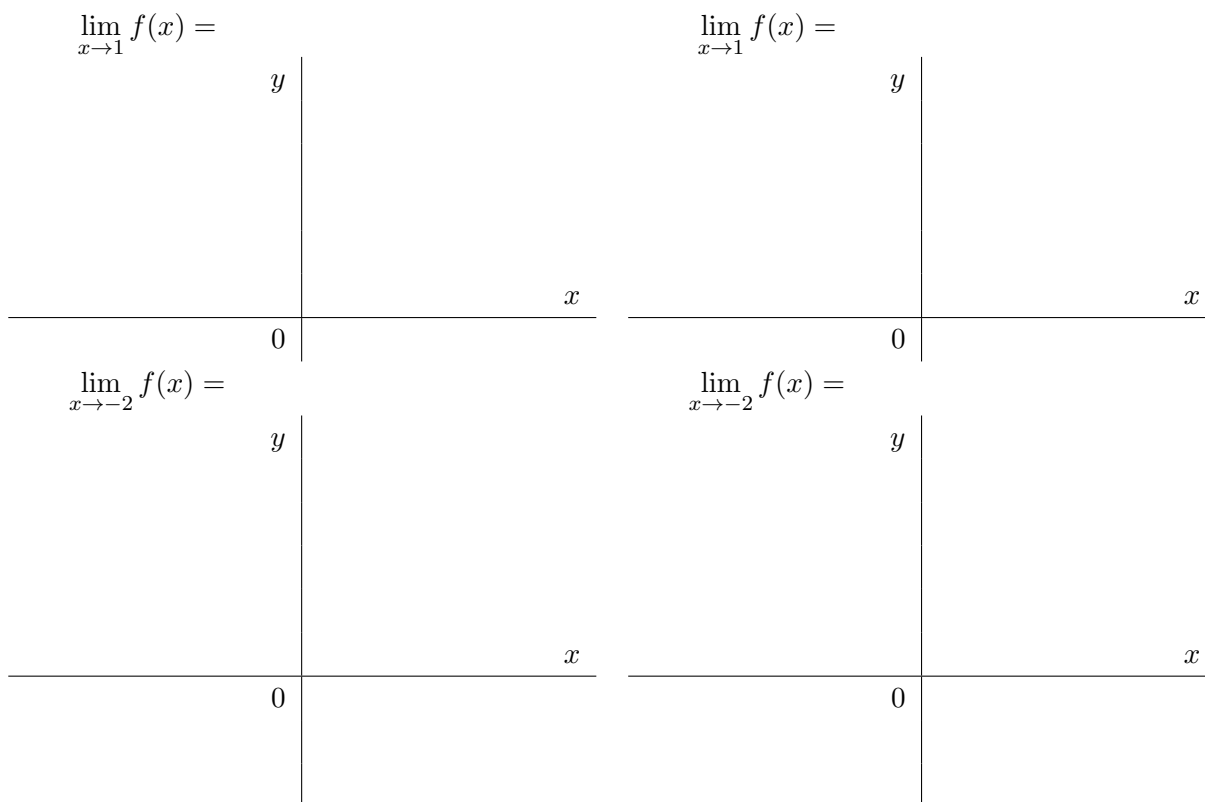
which implies that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

DEFINITION 1.

- If  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$  then  $\lim_{x \rightarrow a} f(x)$  exists and  $\lim_{x \rightarrow a} f(x) = L$ ;
- If  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x) = L$  then  $\lim_{x \rightarrow a} f(x)$  does not exist.

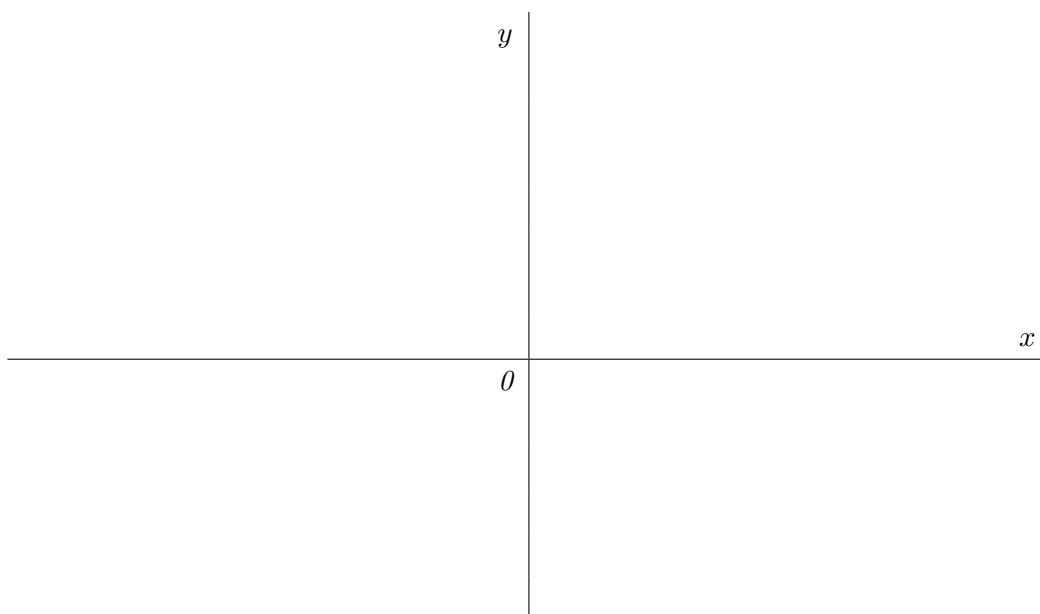




### Limits of piecewise defined function.

EXAMPLE 2. Sketch the graph of the function

$$f(x) = \begin{cases} -3 - x & \text{if } x \leq -2 \\ 2x & \text{if } -2 < x < 2 \\ x^2 - 4x + 3 & \text{if } x \geq 2 \end{cases}$$



Find the limits (using the graph above):

$$\lim_{x \rightarrow 0^-} f(x) =$$

$$\lim_{x \rightarrow -2^-} f(x) =$$

$$\lim_{x \rightarrow 2^-} f(x) =$$

$$\lim_{x \rightarrow 0^+} f(x) =$$

$$\lim_{x \rightarrow -2^+} f(x) =$$

$$\lim_{x \rightarrow 2^+} f(x) =$$

$$\lim_{x \rightarrow 0} f(x) =$$

$$\lim_{x \rightarrow -2} f(x) =$$

$$\lim_{x \rightarrow 2} f(x) =$$

### Limits involving infinity:

DEFINITION 3. The line  $x = a$  is said to be a **vertical asymptote** of the curve  $y = f(x)$  if at least one of the following six statements is true:

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

$$\lim_{x \rightarrow a} f(x) = -\infty$$

REMARK 4. The vertical asymptotes of a rational function come from the zeroes of the denominator.

EXAMPLE 5. Determine the infinite limit:

(a)  $\lim_{x \rightarrow 4^-} \frac{7}{x - 4} =$

(b)  $\lim_{x \rightarrow 4^+} \frac{7}{x - 4} =$

$$(c) \lim_{x \rightarrow 4} \frac{7}{x-4} =$$

$$(d) \lim_{x \rightarrow 0^-} \frac{3-x}{x^4(x+4)} =$$

$$(e) \lim_{x \rightarrow 0^+} \frac{3-x}{x^4(x+4)} =$$

$$(f) \lim_{x \rightarrow 0} \frac{3-x}{x^4(x+4)} =$$

$$(g) \lim_{x \rightarrow \pi^-} \csc x =$$

EXAMPLE 6. Given:  $f(x) = \frac{x-4}{x^2-5x+4}$ .

(a) What are the vertical asymptotes of  $f(x)$ ?

(b) How does  $f(x)$  behave near the asymptotes?