Section 2.2: The Limit of a function

A limit is a way to discuss how the values of a function $f(x)$ behave when x approaches a number a, whether or not $f(a)$ is defined.

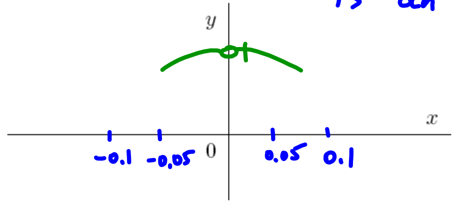
Let's consider the following function:

$$\boxed{f(x) = \frac{\sin x}{x}} \quad (x \text{ in radians}).$$

Note that $f(0) = \dfrac{\sin 0}{0}$ is undefined. However, one can compute the values of $f(x)$ for values of x close to 0.

$f(-x) = \dfrac{\sin(-x)}{-x} = \dfrac{-\sin x}{-x} = \dfrac{\sin x}{x} = f(x) \Rightarrow f(x) = \dfrac{\sin x}{x}$ is an even function

x	$f(x)$
$\pm$ 0.1	0.99833417
$\pm$ 0.05	0.99958339
$\pm$ 0.01	0.99998333
$\pm$ 0.005	0.99999583
$\pm$ 0.001	0.99999983



The table allows us to guess (correctly) that that our function gets closer and closer to 1 as x approaches 0 through positive and negative values. In limit notation it can be written as

$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

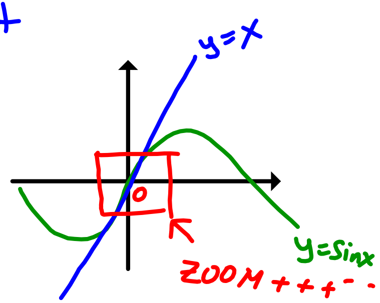
from the left

from the right

which implies that

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1.}$$

Fact



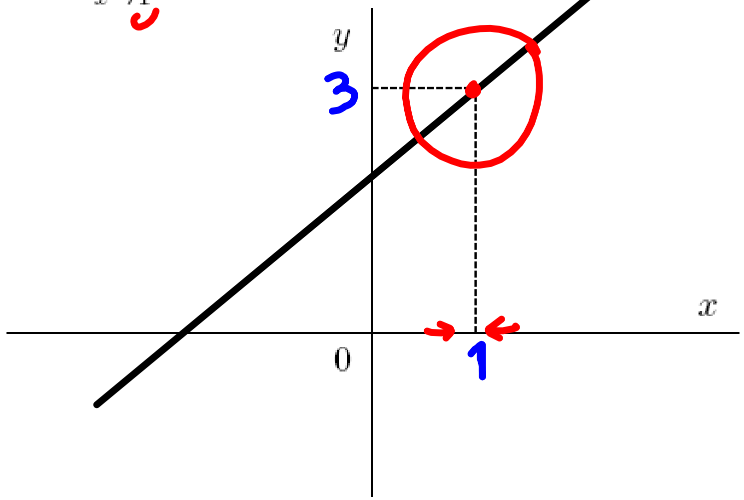
near zero
$\sin x \approx x$
(sinx can be approximated by x

Left (hand) limit — Right (hand) limit — two-sided limit

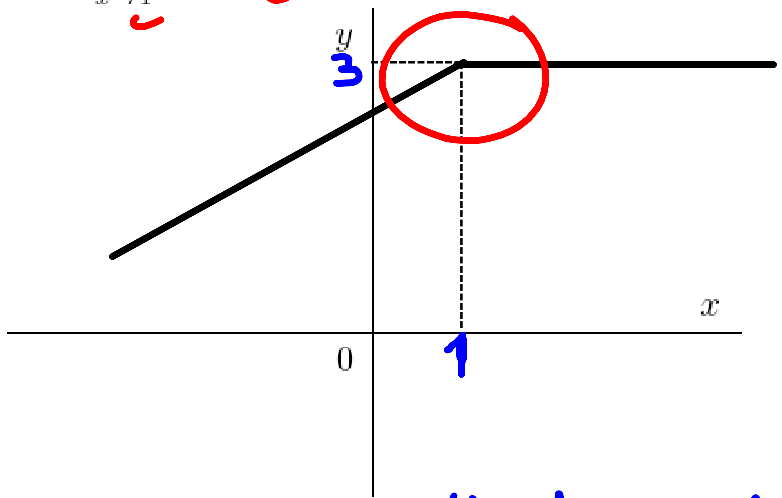
DEFINITION 1.

- *If* $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$ *then* $\lim_{x\to a} f(x)$ *exists and* $\lim_{x\to a} f(x) = L$;
- *If* $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ *then* $\lim_{x\to a} f(x)$ *does not exist.*

$\lim_{x\to 1} f(x) = 3$



$\lim_{x\to 1} f(x) = 3$



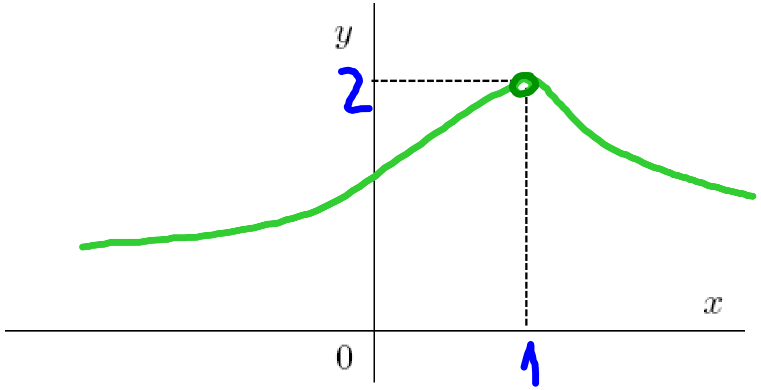
BTW, in both cases $x=1$ belongs to the domain of f and thus $\lim_{x\to 1} f(x) = \underbrace{f(1)}_{\text{direct substitution}} = 3$

Also $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 3.$

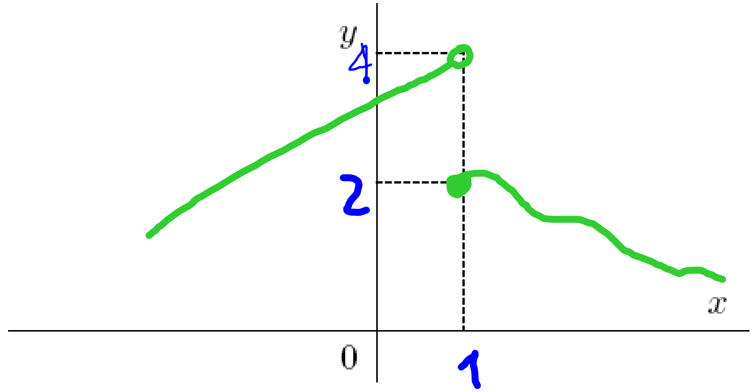
$\lim_{x\to 1^-} f(x) = 4$, but $\lim_{x\to 1^+} f(x) = 2$

$4 \neq 2$

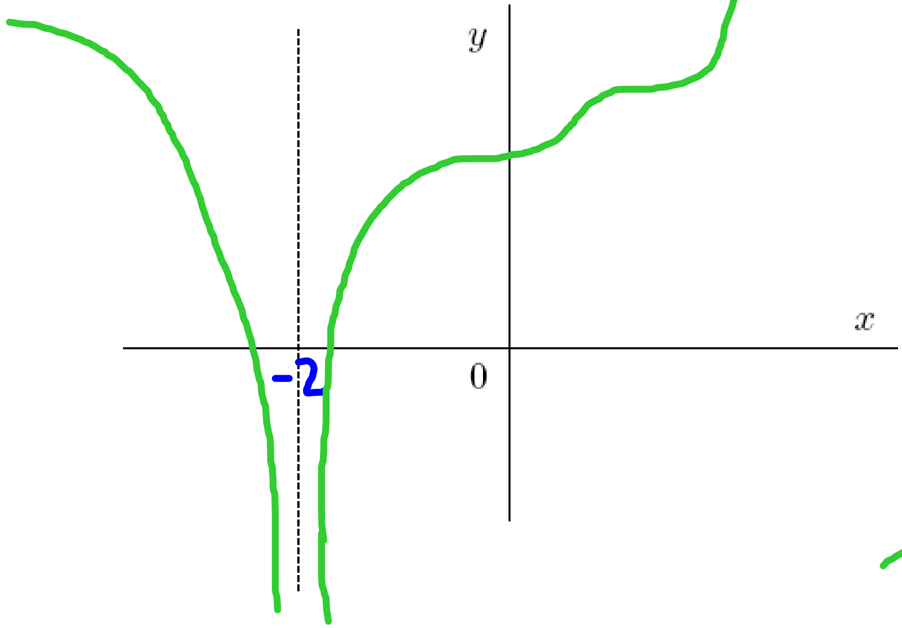
$\lim_{x\to 1} f(x) = 2$



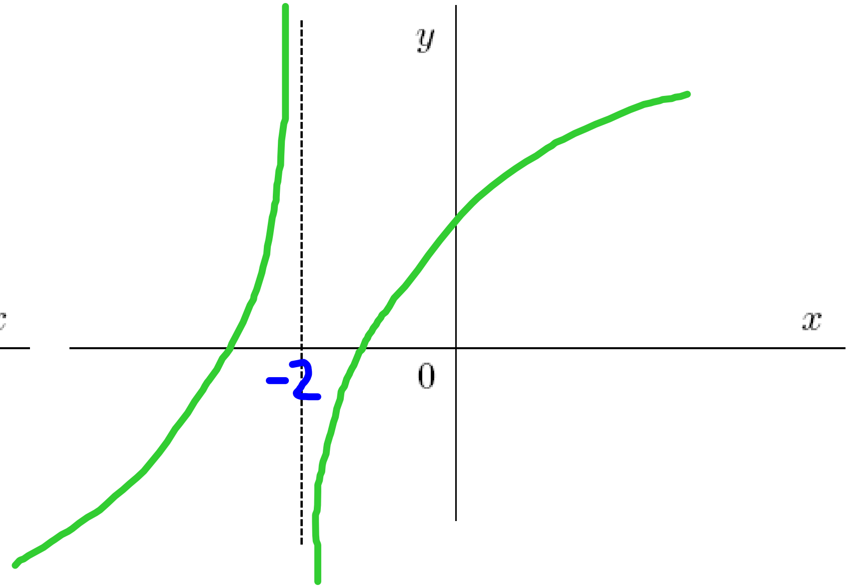
$\lim_{x\to 1} f(x) = \text{DNE}$



$$\lim_{x \to -2} f(x) = -\infty \to \text{DNE}$$



$$\lim_{x \to -2} f(x) = \text{DNE}$$



$$\lim_{x \to -2^-} f(x) = \infty$$

$$\lim_{x \to -2^+} f(x) = -\infty$$

Limits of piecewise defined function.

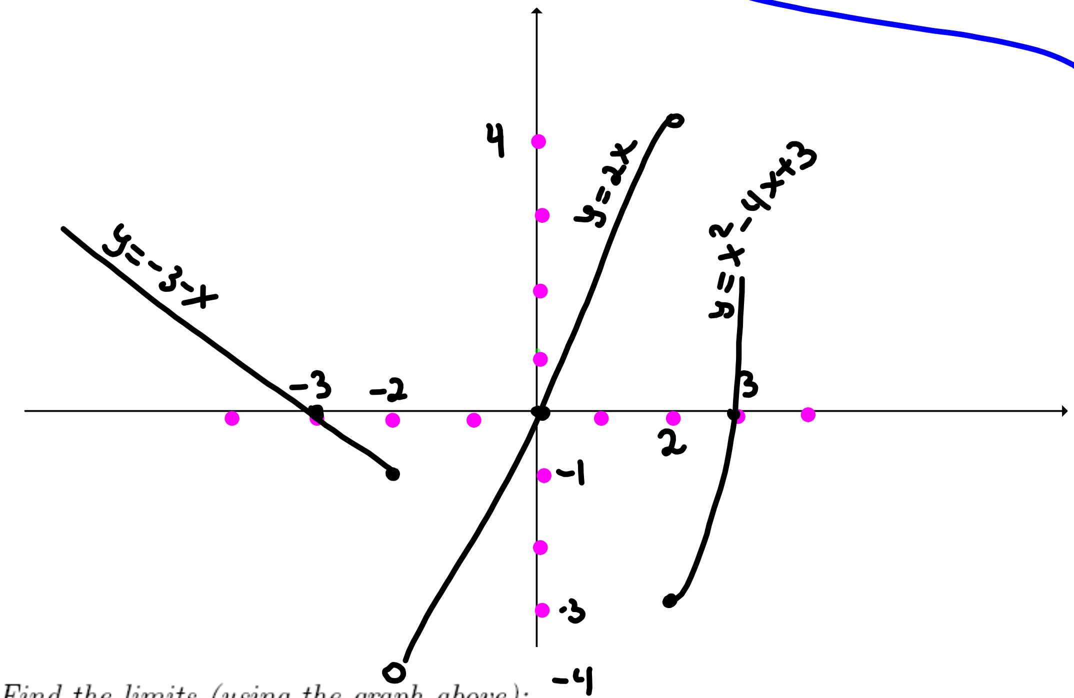
EXAMPLE 2. *Plot the graph of the function*

$$f(x)=\begin{cases} -3-x & \text{if } x\le -2\\ 2x & \text{if } -2<x<2\\ x^2-4x+3 & \text{if } x\ge 2\end{cases}$$

x	y
−3	0
−2	−1

x	y
−2	−4
2	4

x	y
2	−3
3	0



Find the limits (using the graph above):

$\lim\limits_{x\to 0^-} f(x) = f(0)=0$ & $\lim\limits_{x\to 0^+} f(x) = f(0)=0$ $\Rightarrow$ $\lim\limits_{x\to 0} f(x) = 0$ or $\lim\limits_{x\to 0} f(x)=f(0)=0$

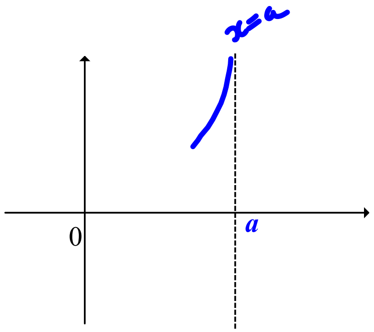
$\lim\limits_{x\to -2^-} f(x) = -1$ & $\lim\limits_{x\to -2^+} f(x) = -4$ $\Rightarrow$ $\lim\limits_{x\to -2} f(x) =$ DNE

$\lim\limits_{x\to 2^-} f(x) = 4$ & $\lim\limits_{x\to 2^+} f(x) = -3$ $\Rightarrow$ $\lim\limits_{x\to 2} f(x) =$ DNE

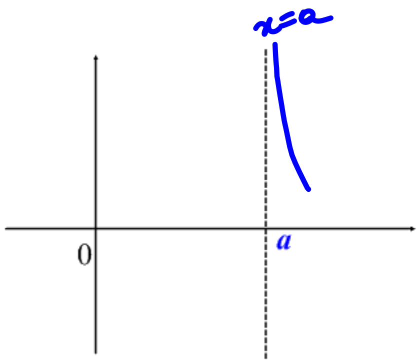
DEFINITION 3. *The line $x = a$ is said to be a* **vertical asymptote** *of the curve $y = f(x)$ if at least one of the following six statements is true:*

graph of the curve

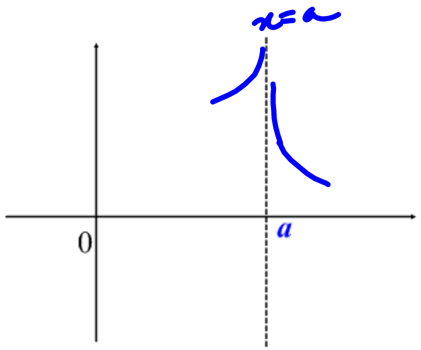
$$\lim_{x \to a^-} f(x) = \infty$$



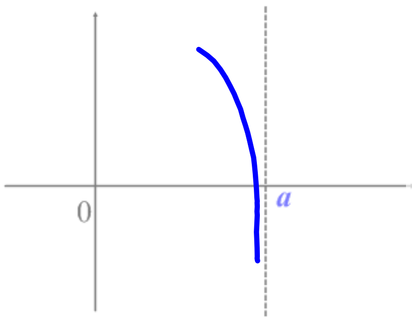
$$\lim_{x \to a^+} f(x) = \infty$$



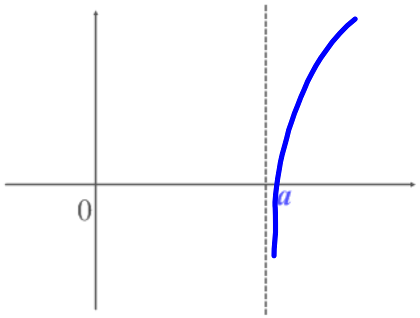
$$\lim_{x \to a} f(x) = \infty$$



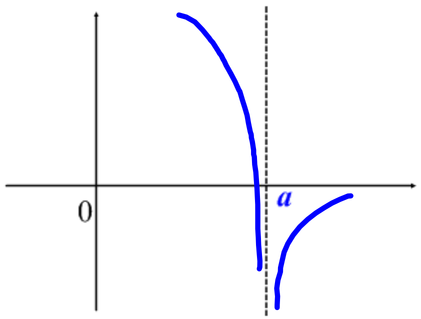
$$\lim_{x \to a^-} f(x) = -\infty$$



$$\lim_{x \to a^+} f(x) = -\infty$$



$$\lim_{x \to a} f(x) = -\infty$$



Note than in all six above situations $f(x)$ is undefined at $x = a$.

Examples of rational functions

$f(x) = \frac{x^7 - 17}{x^2 + 3x + 1}$, $g(x) = \frac{5x}{x^2+1}$, $h(x) = \frac{(x+3)^2}{1}$

ration of two polynomials

REMARK 4. The vertical asymptotes of a rational function come from the zeroes of the denominator.

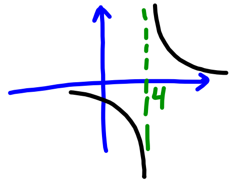
EXAMPLE 5. *Determine the infinite limit:*

$x \to 4^-$ ($x < 4$ and x is "very" close to 4). $\Rightarrow x - 4 < 0 \Rightarrow \frac{7}{x-4} < 0$

(a) $\lim_{x \to 4^-} \frac{7}{x-4} = -\infty$

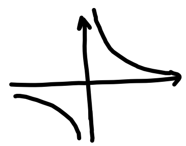
Remark: $x = 4$ is a vertical asymptote for the curve $y = \frac{7}{x-4}$

(b) $\lim_{x \to 4^+} \frac{7}{x-4} = \infty$ $\quad$ $x > 4 \Rightarrow x - 4 > 0 \Rightarrow \frac{7}{x-4} > 0$

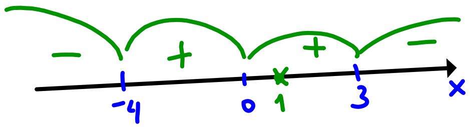


(c) $\lim_{x \to 4} \frac{7}{x-4} =$ DNE

Note $y = \frac{1}{x}$



(d) $\lim_{x \to 0^-} \frac{3-x}{x^4(x+4)} = \infty$



(e) $\lim_{x \to 0^+} \frac{3-x}{x^4(x+4)} = \infty$

Note: $\lim_{x \to 3} \frac{3-x}{x^4(x+4)} = 0$.

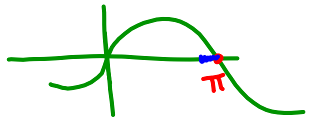
(f) $\lim_{x \to 0} \frac{3-x}{x^4(x+4)} = \infty$

(g) $\lim_{x \to \pi^-} \csc x = \lim_{x \to \pi^-} \frac{1}{\sin x} = \infty$

near $x = \pi$ and $x < \pi$, $\sin x > 0$

$\Downarrow$

$\frac{1}{\sin x} > 0$



EXAMPLE 6. *Given:* $f(x) = \dfrac{x-4}{x^2-5x+4}$.

(a) *What are the vertical asymptotes of $f(x)$?*

If $x \neq 1, 4$, $f(x) = \dfrac{\cancel{x-4}}{\cancel{(x-4)}(x-1)} = \dfrac{1}{x-1}$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \dfrac{1}{x-1} = -\infty \implies x = 1$ is a vertical asymptote.

Remark: Note that $\lim\limits_{x\to 4} f(x) = \lim\limits_{x\to 4} \dfrac{1}{x-1} = \dfrac{1}{3}$ (exists)

Thus the graph of $f(x)$ does not have vertical asymptote through $x=4$.

(b) *How does $f(x)$ behave near the asymptotes?*

$\lim\limits_{x\to 1^-} f(x) = -\infty$

Similarly, $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \dfrac{1}{x-1} = \infty$.

$\lim\limits_{x\to 1} f(x)$ DNE