Section 2.4: The Precise definition of a Limit

DEFINITION 1. *Let $f(x)$ be a function defined for all x in some open interval containing the number a, except possibly at a itself. Then we say that* **the limit of $f(x)$ as x approaches a is L**, *and we write*

$$\lim_{x \to a} f(x) = L,$$

if for every number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad 0 < |x - a| < \delta.$$

REMARK 2. For a limit from the right we need only assume that $f(x)$ is defined on an interval (a, b) extending to the right of a and that the ϵ condition is met for x in an interval $a < x < a + \delta$ extending to the right of a. A similar adjustment must be made for a limit from the left.

A general form of a limit proof

Assume that we are given a positive number ϵ, and we try to prove that we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

There are two things to do:

1. Preliminary analysis of the problem (guessing a value for δ);
2. Proof (showing that the δ works).

Note that *the value of δ is not unique.* Namely, once we have found a value of δ that fulfills the requirements of the definition, then any *smaller* positive number $\delta_1, \delta_1 < \delta$, will also fulfill those requirements.

EXAMPLE 3. *Use the "epsilon-delta" definition to prove that* $\lim_{x\to 4}(3x-1)=11$.

EXAMPLE 4. *Use the "epsilon-delta" definition to prove that* $\lim_{x\to 5} x^2=25$.