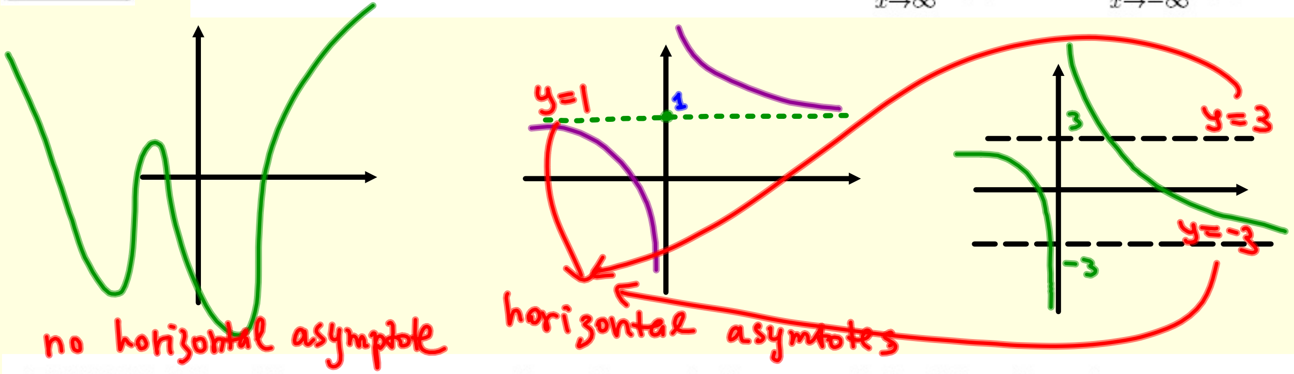
Section 2.6: Limits at infinity: horizontal asymptotes

The end behavior of a function is computed by $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.



THEOREM 1. *If $r > 0$ is a rational number such that x^r is defined for all x, then*

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \quad \textit{and} \quad \lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Corollary: If a is a real number, then $\lim_{x\to\infty} \frac{a}{x^r} = \lim_{x\to-\infty} \frac{a}{x^r} = 0$.

In particular

$$\lim_{x\to\infty} \underbrace{\frac{1}{\sqrt{x}}}_{= \frac{1}{x^{\frac{1}{2}}}} = 0 \qquad \lim_{x\to-\infty} \underbrace{\left(\sqrt[3]{x}\right)^{-1}}_{\frac{1}{x^{\frac{1}{3}}}} = 0$$

$$\sqrt{x^2} = |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$\boxed{\begin{array}{l} x\to\infty \Rightarrow x > 0 \Rightarrow x = \sqrt{x^2} \\ x\to-\infty \Rightarrow x < 0 \Rightarrow x = -\sqrt{x^2} \end{array}}$$

EXAMPLE 2. *Examine the limits at infinity for the following ~~rational~~ functions (the quotient of polynomials):*

$$f(x) = \frac{7x^2+4x-3}{2x^2-12x+16}, \quad g(x) = \frac{9x-4}{7x^2+4x-3}, \quad h(x) = \frac{2x^5-12x^3+16}{x^2-12x+1}$$

Solution: Let's divide the numerator and denominator by the highest power of x that appears and then apply Theorem 1.

- $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(7x^2+4x-3)\frac{1}{x^2}}{(2x^2-12x+16)\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{7x^2}{x^2}+\frac{4x}{x^2}-\frac{3}{x^2}}{\frac{2x^2}{x^2}-\frac{12x}{x^2}+\frac{16}{x^2}}$$

$$= \lim_{x\to\infty} \frac{7+\frac{4}{x}-\frac{3}{x^2}}{2-\frac{12}{x}+\frac{16}{x^2}} = \frac{7+0-0}{2+0-0} = \boxed{\frac{7}{2}}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{7x^2+4x-3}{2x^2-12x+16} = \frac{7}{2}$$

- $$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{(9x-4)\frac{1}{x^2}}{(7x^2+4x-3)\frac{1}{x^2}} =$$

$$= \lim_{x\to\infty} \frac{\frac{9}{x} - \frac{4}{x^2}}{7 + \frac{4}{x} - \frac{3}{x^2}} = \frac{0-0}{7+0-0} = \boxed{0}$$

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{9x-4}{7x^2+4x-3} = 0$$

- $$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{\left(2x^5 - 12x^3 + 16\right)\cdot\frac{1}{x^5}}{\left(x^2 - 12x + 1\right)\frac{1}{x^5}} =$$

$$= \lim_{x\to\infty} \frac{2 - \frac{12}{x^2} + \frac{16}{x^5}}{\frac{1}{x^3} - \frac{12}{x^4} + \frac{1}{x^5}} \overset{\text{DNE}}{=} \infty$$

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{2x^5 - 12x^3 + 16}{x^2 - 12x + 1} = -\infty$$

DEFINITION 3. *If* $\lim_{x\to\infty} f(x) = L$, *or* $\lim_{x\to-\infty} f(x) = L$, *then* $y = L$ *is called a* **horizontal asymptote** *of* $f(x)$.

EXAMPLE 4. *Analyze the rational functions*

$$f(x) = \frac{7x^2+4x-3}{2x^2-12x+16}, \quad g(x) = \frac{9x-4}{7x^2+4x-3}, \quad h(x) = \frac{2x^5-12x^3+16}{x^2-12x+1}$$

from Example 2 for horizontal asymptotes.

function	$f(x)$	$g(x)$	$h(x)$
horizontal asymptotes	$y = \frac{7}{2}$	$y=0$	no

RULES FOR HORIZONTAL ASYMPTOTES of rational functions:

1. If the degree (highest power) of the numerator is larger than the degree of the denominator, then there is no horizontal asymptote (cf. $h(x)$ from Examples 2&4).
2. If the degree of the numerator is smaller than the degree of the denominator, then the horizontal asymptote is at $y = 0$ (or the $x$axis) ((cf. $g(x)$ from Examples 2,4)).
3. If the degree of the numerator is equal to the degree of the denominator, then you must compare the coefficients in front of the terms with the highest power. The horizontal asymptote is the coefficient of the highest power of the numerator divided by the coefficient of the highest power of the denominator. (cf. $f(x)$ from Examples 2,4).

EXAMPLE 5. *Find the equations for all vertical and horizontal asymptotes for*

(a) $f(x) = \dfrac{3x^2 + 2x - 3}{2(x-1)(x+2)}$ rational function

$\boxed{y = \frac{3}{2} \text{ is a horizontal asymptote.}}$

$x=1$ and $x=-2$ are zeroes of the denominator

and $3 \cdot 1^2 + 2 \cdot 1 - 3 \neq 0$, $3 \cdot (-2)^2 + 2(-2) - 3 \neq 0$

(i.e. $x=1$ and $x=-2$ are not zeroes of the numerator)

So, $\boxed{x=1 \text{ and } x=-2 \text{ are vertical asymptotes.}}$

(b) $f(x) = \dfrac{2x}{\sqrt{x^2+5}}$

$x^2 \geq 0$

$x^2 + 5 \geq 0 + 5 > 0$

$x^2 + 5 \neq 0$

$\sqrt{x^2+5} \neq 0$

$f(x)$ is defined for all real values of x.

Thus, it does not have vertical asymptotes.

To find horizontal asymptotes, calculate $\lim\limits_{x \to \pm\infty} f(x)$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+5}} \underset{x=\sqrt{x^2}}{\overset{x>0}{=}}$$

$$= 2 \lim_{x\to\infty} \frac{\sqrt{x^2}}{\sqrt{x^2+5}} = 2 \lim_{x\to\infty} \sqrt{\frac{x^2}{x^2+5}}$$

$$= 2\sqrt{\lim_{x\to\infty} \underbrace{\frac{x^2}{x^2+5}}_{\text{rational function}}} = 2\sqrt{\frac{1}{1}} = \boxed{2}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+5}} = \boxed{-2} \text{ (similarly).}$$

$x < 0 \Rightarrow x = -\sqrt{x^2}$

$y=2$ and $y=-2$ are horizontal asymptotes.

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = (\sqrt{a})^2-(\sqrt{b})^2 = a-b$$

$$(a-b)(a+b) = a^2-b^2$$

EXAMPLE 6. *Compute these limits:*

Multiply and divide by conjugate

(a) $\lim_{x\to\infty}(\sqrt{x^2+3x}-x) =$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+3x}-x)(\sqrt{x^2+3x}+x)}{\sqrt{x^2+3x}+x} = \lim_{x\to\infty} \frac{(\sqrt{x^2+3x})^2 - x^2}{\sqrt{x^2+3x}+x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}+3x-\cancel{x^2}}{\sqrt{x^2+3x}+x} = 3\lim_{x\to\infty} \frac{x\cdot\frac{1}{x}}{(\sqrt{x^2+3x}+x)\frac{1}{x}}$$

$$= 3\lim_{x\to\infty} \frac{1}{\frac{\sqrt{x^2+3x}}{x}+\frac{x}{x}} = 3\lim_{x\to\infty}\frac{1}{\sqrt{\frac{x^2+3x}{x^2}}+1}$$

rational function

$\boxed{\begin{array}{l} x>0 \\ x=\sqrt{x^2} \end{array}}$

$$= 3\cdot\frac{1}{\sqrt{1}+1} = \boxed{\frac{3}{2}}$$

(b) $\underbrace{\lim_{x\to\infty}}\dfrac{\sqrt{3x^4+2x^2+1}}{x(x-1)} = \lim_{x\to\infty}\sqrt{\dfrac{3x^4+2x^2+1}{x^2(x-1)^2}}$

$x>0 \quad x=\sqrt{x^2}$

$x-1>0 \quad x-1=\sqrt{(x-1)^2}$

$\boxed{x(x-1)=\sqrt{x^2(x-1)^2}}$

$= \sqrt{\lim_{x\to\infty}\underbrace{\dfrac{3x^4+2x^2+1}{x^2(x-1)^2}}_{\text{rational function}}} = \sqrt{\dfrac{3}{1}}=\sqrt{3}$

Remark $\quad \lim_{x\to-\infty}\dfrac{\sqrt{3x^4+2x^2+1}}{x(x-1)} = \sqrt{3}$

$$\left.\begin{array}{lll} x<0 & \Rightarrow & x=-\sqrt{x^2} \\ x-1<0 & \Rightarrow & x-1=-\sqrt{(x-1)^2} \end{array}\right\} \Rightarrow \boxed{x(x-1)=\sqrt{x^2(x-1)^2}}$$

(c) $\lim_{x\to\infty}(\sqrt{x^2+4x+5}+x)$ $= \lim_{x\to\infty} \frac{(\sqrt{x^2+4x+5}+x)(\sqrt{x^2+4x+5}-x)}{\sqrt{x^2+4x+5}-x}$

$$= \lim_{x\to\infty} \frac{(\cancel{x^2}+4x+5-\cancel{x^2})\cdot\frac{1}{x}}{(\sqrt{x^2+4x+5}-x)\frac{1}{x}} \overset{x>0}{=} \lim_{x\to\infty} \frac{4+\frac{5}{x}}{\frac{\sqrt{x^2+4x+5}}{\sqrt{x^2}}-1}$$

$$\frac{\sqrt{x^2+4x+5}}{\sqrt{x^2}} = \sqrt{\frac{x^2+4x+5}{x^2}}$$

$$= \lim_{x\to\infty} \frac{4+\frac{5}{x}}{\sqrt{1+\frac{4}{x}+\frac{5}{x^2}}-1} = \infty$$

(d) $\lim_{x\to-\infty}(\sqrt{x^2+4x+5}+x)$

$u=-x \xrightarrow[x\to-\infty]{} \infty$

$\Downarrow$

$x=-u$

$\|$

$$\lim_{u\to\infty}\left(\sqrt{(-u)^2+4(-u)+5}+(-u)\right)=\lim_{u\to\infty}\left(\sqrt{u^2-4u+5}-u\right)$$

$$=\lim_{u\to\infty}\frac{\left(\sqrt{u^2-4u+5}-u\right)\left(\sqrt{u^2-4u+5}+u\right)}{\sqrt{u^2-4u+5}+u}$$

$$=\lim_{u\to\infty}\frac{\left(\cancel{u^2}-4u+5-\cancel{u^2}\right)\cdot\frac{1}{u}}{\left(\sqrt{u^2-4u+5}+u\right)\cdot\frac{1}{u}}=\lim_{u\to\infty}\frac{-4+\frac{5}{u}}{\frac{\sqrt{u^2-4u+5}}{\sqrt{u^2}}+1}$$

$$=\lim_{u\to\infty}\frac{-4+\frac{5}{u}}{\sqrt{1-\frac{4}{u}+\frac{5}{u^2}}+1}=\frac{-4}{\sqrt{1}+1}=-2$$