Section 2.7:Tangents, velocities, and other rates of change.

Consider a curve with equation $y = f(x)$ and points $P(a, f(a))$ and $Q(x, f(x))$ on it. The slope of the secant line PQ (also known as **average rate** or **average velocity**) is

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

DEFINITION 1. *The* **tangent line** *to the curve* $y = f(x)$ *at the point* $P(a, f(a))$ *is the line through* P *with slope*

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

provided that this limit exists.

The slope (1) of tangent line also known as **instantaneous rate of change** or **instantaneous velocity**.

EXAMPLE 2. **(a)** *Find the slope of the tangent line to the graph of* $f(x) = x^2 - 3x - 4$ *at* $(5, 6)$.

(b) *Find the equation of the tangent line to the graph of* $f(x)$ *at* $x = 5$.(Recall that point-slope form for a line through the point (x_1, y_1) with slope m is: $y - y_1 = m(x - x_1)$.)

DEFINITION 3. *If* $\mathbf{r}(t)$ *is a vector function then the* **tangent vector v** *at* $t = a$ *is found by*

$$\mathbf{v} = \lim_{t \to a} \frac{1}{t - a} \left[\mathbf{r}(t) - \mathbf{r}(a)\right].$$

EXAMPLE 4. *Given curve* $\mathbf{r}(t) = \langle 2t, 10t - t^2 \rangle$.

(a) *Find a vector tangent to the curve at the point* $(4, 16)$.

(b) *Find parametric equations of the tangent line to* $\mathbf{r}(t)$ *at* $t = 2$.

(c) *Find a cartesian equation of this tangent line.*

Velocities. Denote by $f(t)$ the position of an object at time t.
The **Average Velocity** of the object from $t = a$ to $t = b$ is

$$\frac{f(b) - f(a)}{b - a}.$$

The **Instantaneous Velocity** of the object at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

(Compare this formula with (1) by substitution $h = x - a$.)

EXAMPLE 5. *The position (in meters) of an object moving in a straight path is given by*

$$s(t) = t^2 - 2t + 8,$$

where t is measured in seconds.

(a) *Find the average velocity over the time interval $[4, 5]$.*

(b) *Find the instantaneous velocity at time $t = 4$.*

Other Rates of Change:

The **Average Rate** of change of function $f(x)$ from $x = a$ to $x = b$ is

$$\frac{f(b) - f(a)}{b - a}.$$

The **Instantaneous Rate of Change** of $f(x)$ at $x = a$ is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$