3.11: Differentials; Linear Approximations

Differentials: If x changes from x_1 to x_2, then the *change in* x is

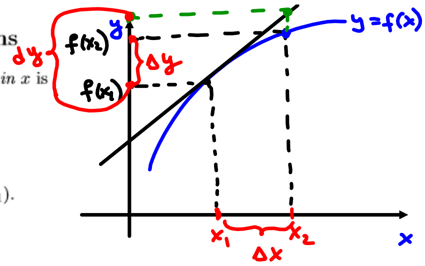
$$\Delta x = x_2 - x_1.$$

If $y = f(x)$ then corresponding *change in* y is

$$\Delta y = f(x_2) - f(x_1).$$

$\frac{\Delta y}{\Delta x} = f'(x)$

$\Delta y = f'(x)\,\Delta x$



DEFINITION 1. *Let $y = f(x)$, where f is a differentiable function. Then the differential dx is an independent variable (i.e. dx can be given the value of any real number). The differential dy is then defined in terms of dx by the equation*

$$\boxed{dy = f'(x)dx.}$$

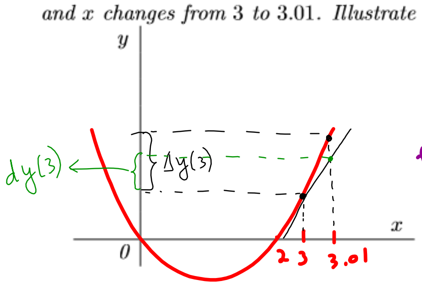
$df(x) = f'(x)\,dx$

EXAMPLE 2. *Compare the values of Δy and dy if*

$\Delta x = x_2 - x_1 = 0.01$

$$y = f(x) = x^2 - 2x$$

and x changes from 3 (x_1) to 3.01 (x_2). Illustrate these quantities graphically.



$\Delta y(3) = f(3.01) - f(3)$

$= 3.01^2 - 2\cdot 3.01 - (3^2 - 2\cdot 3) = \boxed{0.0401}$

$f'(x) = 2x - 2 \Rightarrow f'(3) = 2\cdot 3 - 2 = 4$

$dy = f'(3)\,\Delta x = 4 \cdot 0.01 = \boxed{0.04}$

REMARK 3. Notice that dy was easier to compute than Δy. For more complicated functions (for example $y = \cos x$) it may be impossible to compute Δy exactly.

CONCLUSION: $\boxed{\Delta y \approx dy}$ provided we keep Δx small.

$\Delta x = dx$

EXAMPLE 4. *A sphere was measured and its radius was found to be 15 inches with a possible error of no more that 0.02 inches. If we use this value of the radius, find the following quantities:*

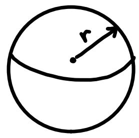
(a) *the maximum possible error in the volume of the sphere;*

$r = 15, \quad \Delta r = 0.02$

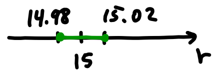
$V(r) = \frac{4}{3}\pi r^3 \Rightarrow V'(r) = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$

$\Delta V \approx dV = V'(r)\,dr = V'(r)\,\Delta r$

$\Delta V(15) = V'(15)\,\Delta r = 4\pi \cdot 15^2 \cdot 0.02 \approx 56.5487 \text{ in.}^3$



$r = 15'' \pm 0.02''$



$V(15) = \frac{4}{3}\pi 15^3 \approx 14131.17$

$V \approx 14131.17 \pm 56.5487$

(b) *the relative error in the radius of the sphere;*

$\frac{\Delta r}{r} = \frac{0.02}{15} \approx 0.0013333$

(c) *the relative error in the volume of the sphere.*

$\frac{\Delta V}{V} = \frac{V'\Delta r}{V} = \frac{\cancel{4\pi} \cancel{r^2}\,\Delta r}{\cancel{\frac{4}{3}\pi} \cancel{r^3}} = 3\left(\frac{\Delta r}{r}\right) \approx 3 \cdot 0.0013333 = 0.003999$

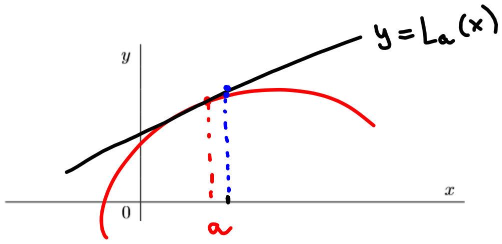
Linear Approximation: The function

$$\boxed{L_a(x) = f(a) + f'(a)(x-a)}$$

(whose graph, $y = L_a(x)$, is the tangent line to the curve $y = f(x)$ at the points $(a, f(a))$) is called the **linearization of f at a.** The approximation $\boxed{f(x) \approx L_a(x)}$ or

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)} \qquad (1)$$

is called the **linear approximation** or **tangent line approximation** of f at a.



Rewriting the formula (1) when $x = a + \Delta x$, we get the following "approximation by differentials" formula:

$f(x) \approx f(a) + f'(a)(x-a)$

$x = a + \Delta x \Rightarrow x - a = \Delta x$

$f(a + \Delta x) \approx f(a) + \underbrace{f'(a)\,\Delta x}_{df(a)}$

EXAMPLE 5. *Determine the linaerization and the linear approximation for* $\sin x$ *at* $a = 0$. *Then find* $\sin(0.000007)$.

$f(x) = \sin x$, $a = 0$.

$$L_0(x) = f(0) + f'(0)(x-0)$$

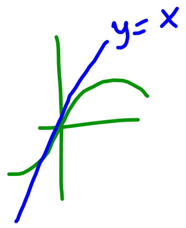
$$f(0) = \sin 0 = 0$$

$$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$$

$\Rightarrow$

$$\Rightarrow L_0(x) = 0 + 1 \cdot x$$

$$L_0(x) = x$$



$\Downarrow$

$\sin x \approx x$ near zero.

$$\sin(0.000007) \approx 0.000007$$

EXAMPLE 6. *Use a linear approximation (or differentials) or to find an approximate value for* $\sin 31°$.

$$f(x) \approx f(a) + f'(a)(x-a)$$

$x = 31°$

$f(x) = \sin x$

$a = 30° = \frac{\pi}{6}$

$f(a) = f(\frac{\pi}{6}) = \sin\frac{\pi}{6} = \boxed{\frac{1}{2}}$

$f'(x) = (\sin x)' = \cos x$

$f'(a) = f'(\pi/6) = \cos \pi/6 = \boxed{\frac{\sqrt{3}}{2}}$

$x - a = 31° - 30° = 1° = \boxed{\frac{\pi}{180}}$

$$\sin 31° \approx \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{180} \approx 0.155115$$

Using calculator: $\sin 31° \approx 0.515038$

EXAMPLE 7. *Use a linear approximation (or differentials) to find an approximate value for* $\sqrt[4]{0.98}$.

$f(x) = \sqrt[4]{x}$

$a = 1$

$x = 0.98$

$f'(x) = (x^{\frac{1}{4}})' = \frac{1}{4} x^{-3/4}$

$f'(a) = f'(1) = \frac{1}{4}$

$x - a = 0.98 - 1 = -0.02$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$f(0.98) = \sqrt[4]{0.98} \approx 1 + \frac{1}{4}(-0.02)$$

$$= 1 - 0.005 = \boxed{0.995}$$

Using calculator: $\sqrt[4]{0.98} \approx 0.99496$