3.7: Derivatives of the vector functions

EXAMPLE 1. *Sketch the curve* $\mathbf{r(t)}$ *and indicate with arrow the direction in which* t *increases if*

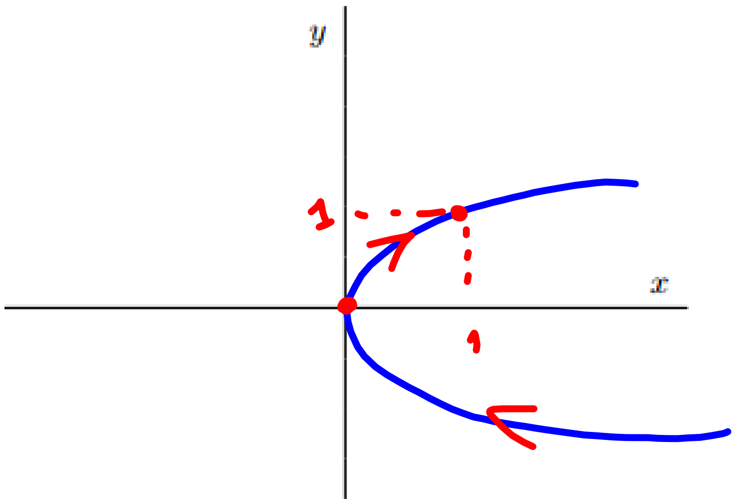
(a) $\mathbf{r(t)} = \langle t^2, t \rangle$

$x = t^2$, $y = t$

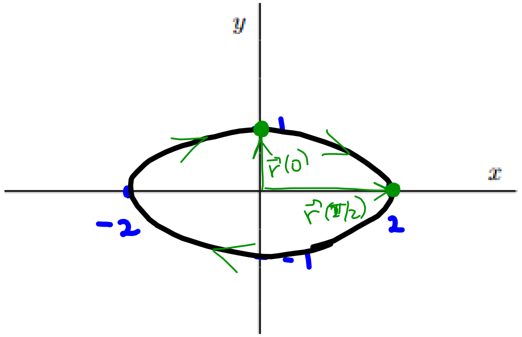
$\boxed{x = y^2}$

$\vec{r}(0) = \langle 0, 0 \rangle$

$\vec{r}(1) = \langle 1, 1 \rangle$



(b) $\mathbf{r(t)} = \langle 2\sin t, \cos t \rangle$



$x = 2\sin t, \quad y = \cos t$

$\sin t = \frac{x}{2}, \quad \cos t = y$

$\sin^2 t + \cos^2 t = 1$

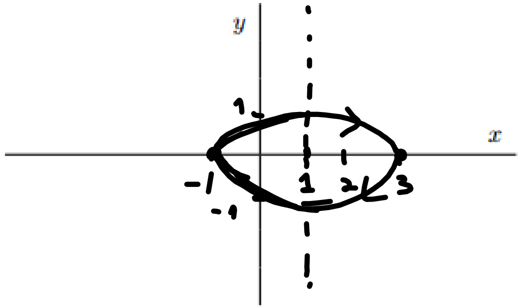
$\left(\frac{x}{2}\right)^2 + y^2 = 1$

$\frac{x^2}{4} + y^2 = 1$ ellipse

$\vec{r}(0) = \langle 0, 1 \rangle$

$\vec{r}\left(\frac{\pi}{2}\right) = \langle 2, 0 \rangle$

(c) $\mathbf{r(t)} = \langle 1 + 2\sin t, \cos t \rangle$



see (b)

$x = 1 + 2\sin t$, $y = \cos t$

$\sin t = \frac{x-1}{2}$, $\cos t = y$

Use $\sin^2 t + \cos^2 t = 1$

$$\frac{(x-1)^2}{4} + y^2 = 1.$$

$y = 0$ $\quad (x-1)^2 = 4$

$x = -1$

$x = 3$

Conclusion:

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

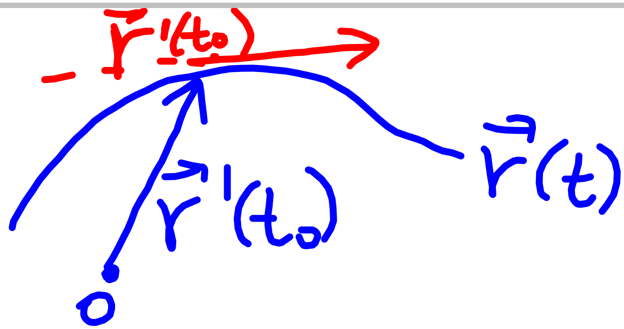
in param. form can be represented as:

$$\vec{r}(t) = \langle x_0 + a\cos t, \ y_0 + b\sin t \rangle$$

DEFINITION 2. *If* $\mathbf{r}(\mathbf{t}) = \langle x(t), y(t) \rangle$ *is a vector function, then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

if both $x'(t), y'(t)$ *exist.*

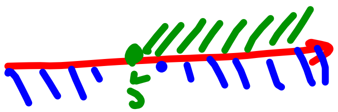


EXAMPLE 3. *If* $\mathbf{r}(\mathbf{t}) = \langle t^2, \sqrt{t-5} \rangle$ *find the domain of* $\mathbf{r}(t)$ *and* $\mathbf{r}'(t)$.

$x(t) = t^2$, $y(t) = \sqrt{t-5}$

Domain of $x(t)$: $(-\infty, \infty)$

Domain of $y(t)$: $t-5 \geq 0 \Rightarrow t \geq 5 \Rightarrow [5, \infty)$ $\Big\} \Rightarrow$



Domain of $\vec{r}(t)$: $(-\infty, \infty) \cap [5, \infty) = \boxed{[5, \infty)}$

Find $\vec{r}'(t)$;

$$\vec{r}'(t) = \langle x'(t), y'(t) \rangle = \left\langle 2t, \frac{1}{2\sqrt{t-5}} \right\rangle$$

Domain of $x'(t)$: $(-\infty, \infty)$

Domain of $y'(t)$: $(5, \infty)$ $\Big\} \Rightarrow$

Domain of $\vec{r}'(t)$: $(5, \infty)$.

DEFINITION 4. *If* $(r)(t) = \langle x(t), y(t)\rangle$ *is a vector function representing the position of a particle at time* t*, then*

- **instantaneous velocity** *at time* t *is* $\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle$
- **instantaneous speed** *at time* t *is* $|\mathbf{r}'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$

EXAMPLE 5. *The vector function* $\mathbf{r}(t) = \left\langle t, \sqrt{t^2+9}\right\rangle$ *represents the position of a particle at time* t*. Find the velocity and speed of the particle at time* $t = 4$.

$$v(t) = \vec{r}\,'(t) = \left\langle \frac{d}{dt}(t), \frac{d}{dt}\left(\sqrt{\underbrace{t^2+9}_{u}}\right) \right\rangle$$

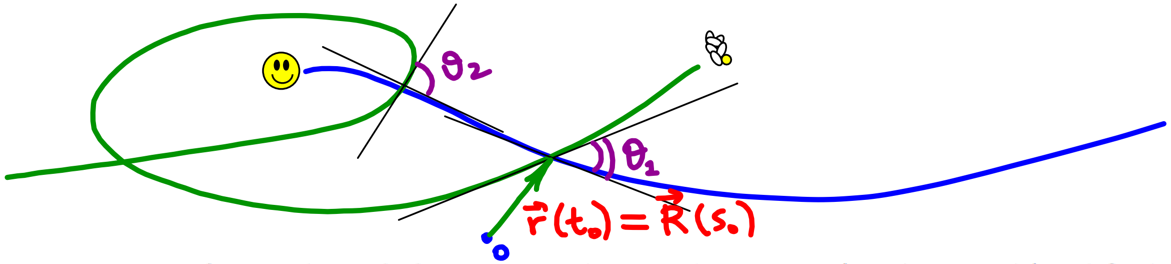
$$= \left\langle 1, \frac{1\cdot(t^2+9)'}{2\sqrt{t^2+9}} \right\rangle = \left\langle 1, \frac{2t}{2\sqrt{t^2+9}} \right\rangle$$

$$v(t) = \left\langle 1, \frac{t}{\sqrt{t^2+9}} \right\rangle \Rightarrow v(4) = \left\langle 1, \frac{4}{\sqrt{4^2+9}} \right\rangle$$

$$= \boxed{\left\langle 1, \frac{4}{5} \right\rangle}$$

$$\text{speed} = |v(4)| = \left|\left\langle 1, \frac{4}{5}\right\rangle\right| =$$

$$= \sqrt{1^2 + \frac{4^2}{5^2}} = \frac{\sqrt{25+16}}{\sqrt{5^2}} = \frac{\sqrt{41}}{5} \text{ units}$$



DEFINITION 6. *The* **angle between two intersecting curves** *(curvilinear angle) is defined to be the angle between the tangent lines at the point of intersection.*

$0 \le \theta \le \frac{\pi}{2}$

EXAMPLE 7. *Given two curves traced by*

$$\mathbf{r}(t) = \langle 1 + t, 3 + t^2 \rangle, \qquad \mathbf{R}(s) = \langle 2 - s, s^2 \rangle .$$

(a) *At what point do the curves intersect?*

$$\vec{r}(t) = \vec{R}(s)$$

$$\langle 1+t, 3+t^2 \rangle = \langle 2-s, s^2 \rangle$$

$$\begin{cases} 1+t = 2-s \\ 3+t^2 = s^2 \end{cases} \Rightarrow t = 1-s$$

$$3 + (1-s)^2 = s^2$$

$$3 + 1 - 2s + \cancel{s^2} = \cancel{s^2} \Rightarrow 2s = 4 \Rightarrow s = 2$$

$$t = 1 - s = -1.$$

Intersection point : $\vec{r}(-1) = \langle 1-1, 3+(-1)^2 \rangle = \langle 0, 4 \rangle$
is $(0,4)$.

Note that $\vec{R}(2) = \langle 0, 4 \rangle$.

(b) *Find the angle between the curves.* = angle between tangent lines at $(0,4)$

= angle between $\vec{r}\,'(-1)$ and $\vec{R}\,'(2)$ = angle between $\langle 1,-2\rangle$ and $\langle -1,4\rangle$

$$\boxed{\begin{aligned}\vec{r}\,'(t) &= \langle 1, 2t\rangle\\ \vec{R}\,'(s) &= \langle -1, 2s\rangle\\ \vec{r}\,'(-1) &= \langle 1, -2\rangle\\ \vec{R}\,'(2) &= \langle -1, 4\rangle\end{aligned}}$$

$$\cos\theta = \frac{\langle 1,-2\rangle\cdot\langle -1,4\rangle}{|\langle 1,-2\rangle|\cdot\langle -1,4\rangle}$$

$$= \frac{1\cdot(-1) + (-2)\cdot 4}{\sqrt{1^2+(-2)^2}\cdot\sqrt{(-1)^2+4^2}}$$

$$= \frac{-1-8}{\sqrt{5}\,\sqrt{17}} = -\frac{9}{\sqrt{85}}$$

$$\theta = \cos^{-1}\left(-\frac{9}{\sqrt{85}}\right) = \ldots$$