4.1: Exponential functions and their derivatives

An **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant. It is defined is the following manner:

- If $x = n$, a positive integer, then $a^n = \underbrace{a \cdot a \cdot \quad \cdots \quad \cdot a}_{n \text{ factors}}$
- If $x = 0$ then $a^0 = 1$.
- If $x = -n$, n is a positive integer, then $a^{-n} = \frac{1}{a^n}$.
- If x is a rational number, $x = \frac{p}{q}$, with p and q integers and $q > 0$, then

$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

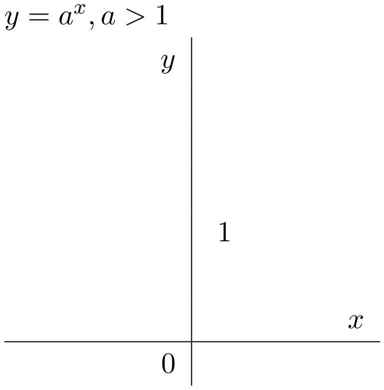
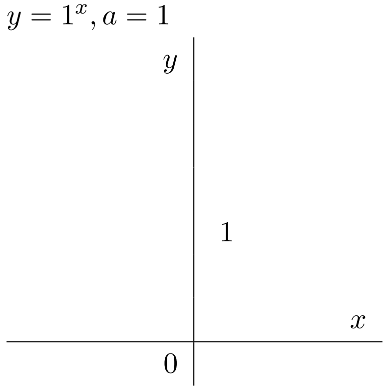
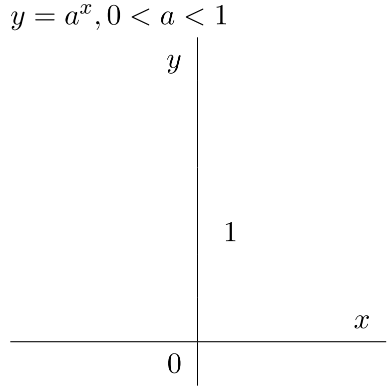
- If x is an irrational number then we define

$$a^x = \lim_{r \to x} a^r$$

where r is a rational number.

It can be shown that this definition uniquely specifies a^x and makes the function $f(x) = a^x$ continuous.

There are basically 3 kinds of exponential functions $y = a^x$:

Exponential growth	Constant	Exponential Decay
$y = a^x, a > 1$	$y = 1^x, a = 1$	$y = a^x, 0 < a < 1$
		
Domain: Range: $\lim_{x\to\infty} a^x =$ $\lim_{x\to-\infty} a^x =$ horizontal asymptote:		Domain: Range: $\lim_{x\to\infty} a^x =$ $\lim_{x\to-\infty} a^x =$ horizontal asymptote:

PROPERTIES OF THE EXPONENTIAL FUNCTION:

If $a, b > 0$ and x, y are real then

$$1.\ a^{x+y} = a^x a^y \quad 2.\ a^{x-y} = \frac{a^x}{a^y} \quad 3.\ (a^x)^y = a^{xy} \quad 4.\ (ab)^x = a^x b^x.$$

EXAMPLE 1. *Find the limit:*

(a) $\lim\limits_{x\to\infty} (4^{-x} - 3)$

(b) $\lim\limits_{x\to\infty} \left(\frac{\pi}{7}\right)^x$

(c) $\lim\limits_{x\to-\infty} \left(\pi^2 - 7\right)^x$

(d) $\lim\limits_{x\to 3^+} \left(\frac{1}{7}\right)^{\frac{x}{x-3}}$

There are in fact a variety of ways to define e. Here are two of them:

1. $e = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$

2. e is the unique positive number for which $\lim\limits_{h\to 0} \frac{e^h - 1}{h} = 1.$

It can be also shown that $e \approx 2.71828$.

EXAMPLE 2. *Find the limit:*

(a) $\lim\limits_{x\to 1^+} e^{\frac{4}{x-1}}$

(b) $\lim\limits_{x \to 1^-} e^{\frac{4}{x-1}}$

(c) $\lim\limits_{x \to \infty} \dfrac{e^{5x} - e^{-5x}}{e^{5x} + e^{-5x}}$

Derivative of exponential function.

EXAMPLE 3. *Find the derivative of* $f(x) = e^x$.

CONCLUSIONS:

1. e^x is differentiable function.
2. If $u(x)$ is a differentiable function then by Chain Rule: $\dfrac{\mathrm{d}}{\mathrm{d}x} e^{u(x)} = e^u \dfrac{\mathrm{d}u}{\mathrm{d}x}$.

EXAMPLE 4. *Find the derivative of the function* $f(x) = e^{x \sin x}$.

EXAMPLE 5. *For what value(s) of* A *does the function* $y = e^{Ax}$ *satisfy the equation* $y'' + 2y' - 8y = 0$*?*