4.2:Inverse Functions

DEFINITION 1. *A function of domain X is said to be a* **one-to-one** *function if no two elements of X have the same image, i.e.*

if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Equivalently, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Horizontal line test:A function if one-to-one is and only if no horizontal line intersects its graph more once.

EXAMPLE 2. *Are the following functions one-to-one?*

$$f(x) = x^3, \quad g(x) = \sqrt{x} + 3, \quad h(x) = x^2, \quad u(x) = |x|, \quad w(x) = \sin x, \quad F(x) = -x^2 + x + 1$$

EXAMPLE 3. *Prove that* $f(x) = \dfrac{x-3}{x+3}$ *is one-to-one.*

EXAMPLE 4. *How we can restrict the domain of $f(x) = \sin x$ to make it one-to-one?*

DEFINITION 5. *Let f be a one-to-one function with domain X and range Y. Then the inverse function f^{-1} has the domain Y and range X and is defined for any y in Y by*

$$f^{-1}(y) = x \Leftrightarrow f(x) = y.$$

REMARK 6. Reversing roles of x and y in the last formula we get:

$$f^{-1}(x) = y \Leftrightarrow f(y) = x.$$

REMARK 7. If $y = f(x)$ is one-to-one function with the domain X and the range Y then

for every x in X $\quad f^{-1}(f(x)) =$

and

for every x in Y $\quad f(f^{-1}(x)) =$

CAUTION: $f^{-1}(x)$ does NOT mean $\frac{1}{f(x)}$.

TO FIND THE INVERSE FUNCTION OF A ONE-TO-ONE FUNCTION f:

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

EXAMPLE 8. *(cf. Example3) Find the inverse function of* $f(x) = \dfrac{x-3}{x+3}$.

EXAMPLE 9. *Given* $f(x) = x^2 + x$, $x \geq \frac{1}{2}$. *Find the inverse function of* f.

FACT: The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

THEOREM 10. *If* f *is a one-to-one differentiable function with inverse function* $g = f^{-1}$ *and* $f'(g(a)) \neq 0$, *then the inverse function is differentiable at* a *and*

$$g'(a) = \frac{1}{f'(g(a))}.$$

Proof.

EXAMPLE 11. *Suppose that g is the inverse function of f and $f(4) = 5$, $f'(4) = 7$. Find $g'(5)$.*

EXAMPLE 12. *Suppose that g is inverse of f. Find $g'(a)$ where*

(a) $f(x) = \sqrt{x^3 + x^2 + x + 1}$, $a = 2$

(b) $f(x) = \dfrac{2x - 3}{x + 3}$, $a = \dfrac{1}{2}$.

(c) $f(x) = 4 + 3x + e^{3(x-1)}$, $a = 8$.