4.3:Logarithmic Functions

$a>0 \Rightarrow a^x>0$

DEFINITION 1. *The exponential function $f(x)=a^x$ with $a\neq 1$ is a one-to-one function. The inverse of this function, called the* **logarithmic function** *with* **base** a, *is denoted by $f^{-1}(x)=\log_a x$.*

Namely,

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x.$$

(OR $\log_a y = x \Leftrightarrow a^x = y$)

In other words, if $x>0$ then $\log_a(x)$ is the exponent to which the base a must be raised to give x.

EXAMPLE 2. *Evaluate*

(a) $\log_2 16 = 4$ $\quad (2^4=16)$

(b) $\log_3 \frac{1}{81} = -4$ $\quad \left(3^{-4} = \left(\frac{1}{3}\right)^4 = \frac{1}{81}\right)$

(c) $\log_{125} 5 = \frac{1}{3}$ $\quad \left(125 = 5^3 \Rightarrow \sqrt[3]{125} = 5 \Rightarrow 125^{\frac{1}{3}} = 5\right)$

(d) $\log_1 125$ undefined $\quad (a\neq 1)$

(e) $\log_a 1 = 0$ $\quad (a^0 = 1)$

CANCELLATION RULES:

- $\log_a a^x = x$ for all $x\in\mathbb{R}$
- $a^{\log_a x} = x$ for $x>0$.

$f(x) = \log_a x \quad , \quad f^{-1}(x) = a^x$

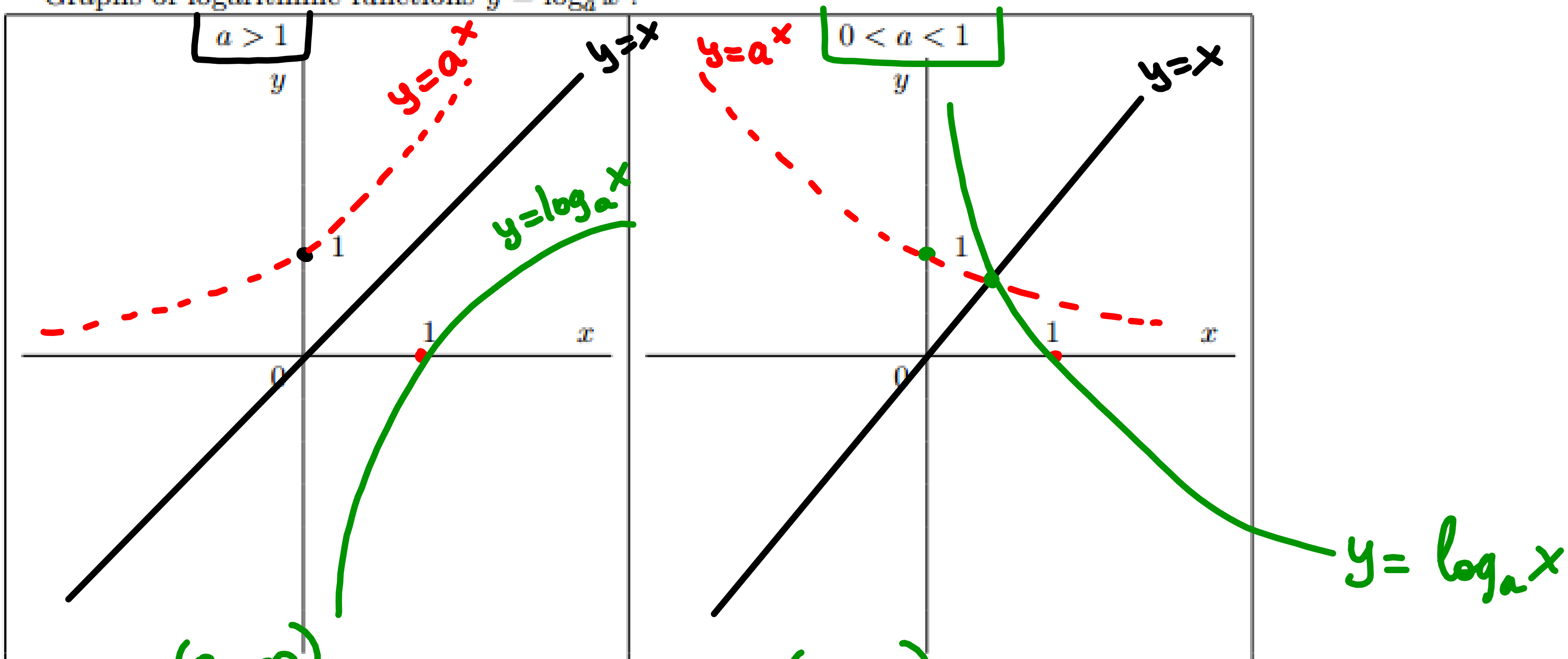
$f(f^{-1}(x)) = x$

$f^{-1}(f(x)) = x$

$\log_2 2^4 = 4$

$2^{\log_2 4} = 4$

Graphs of logarithmic functions $y = \log_a x$:



$a > 1$	$0 < a < 1$
Domain: $(0, \infty)$	Domain: $(0, \infty)$
Range: $\mathbb{R}$	Range: $\mathbb{R}$
$\lim\limits_{x\to\infty} \log_a x = \infty$	$\lim\limits_{x\to\infty} \log_a x = -\infty$
$\lim\limits_{x\to 0^+} \log_a x = -\infty$	$\lim\limits_{x\to 0^+} \log_a x = \infty$
Vertical asymptote: $x=0$	Vertical asymptote: $x=0$
Horizontal asymptote: none	Horizontal asymptote: none

Properties: Assume that $a \neq 1$ and $x, y > 0$.

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^y) = y \log_a x$$

$$\log_7 6 = \log_7 2 + \log_7 3$$

$$\log_7 6 = \log_7 24 - \log_7 4$$

$$2\log_a x = \log_a x + \log_a x = \log_a x^2$$

$$\log_7 81 = 4\log_7 3$$

In particular,

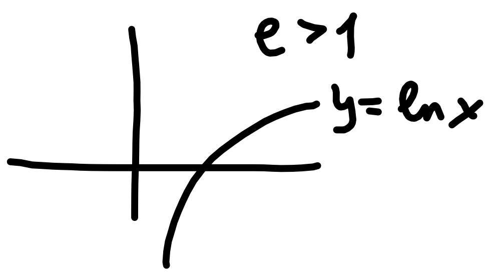
$$\log_a \sqrt{x} = \frac{1}{2}\log_a x$$

$$\log_a \sqrt[n]{x} = \frac{1}{n}\log_a x$$

Notation: *Common Logarithm:* $\log x = \log_{10} x$. (Thus, $\log x = y \quad \Leftrightarrow \quad 10^y = x$.)
Natural Logarithm: $\ln(x) = \log_e(x)$. (Thus, $\underline{\ln x = y \quad \Leftrightarrow \quad e^y = x}$.)

Properties of the natural logarithms:

- $\ln(e^x) = \log_e e^x = x$ } cancellation laws
- $e^{\ln x} = x$
- $\ln e = 1 \quad (e^1 = e)$
- $\log_a x = \dfrac{\ln x}{\ln a}$, where $a > 0$ and $a \neq 1$; Base change formula.
- $\lim_{x\to\infty} \ln x = \infty$
- $\lim_{x\to 0^+} \ln x = -\infty$



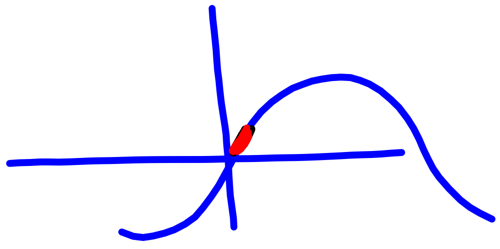
EXAMPLE 3. *Find each limit:*

(a) $\lim\limits_{x\to\infty} \ln(x^2-x) = \lim\limits_{u\to\infty} \ln u = \infty$

$$\boxed{u = x^2 - x = \underbrace{x(x-1)}_{>0} \xrightarrow[x\to\infty]{} \infty}$$

(b) $\lim\limits_{x\to 0^+} \log(\sin x) = \lim\limits_{u\to 0^+} \log u = -\infty$

$$u = \underbrace{\sin x}_{>0} \xrightarrow[x\to 0^+]{} 0^+$$



EXAMPLE 4. *Find the domain of* $f(x) = \ln(x^3 - x)$.

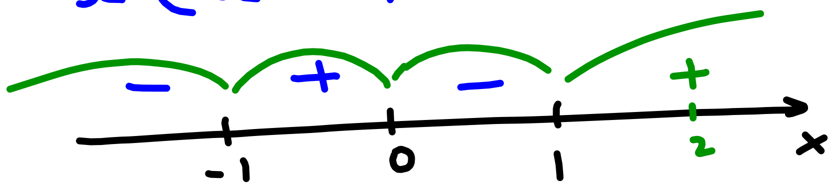
$\ln u$

$u > 0$

$x^3 - x > 0$

$x(x^2 - 1) > 0$

$x(x+1)(x-1) > 0$



$-1 < x < 0$ OR $x > 1$

$\mathrm{Dom}(f) = (-1, 0) \cup (1, \infty)$

EXAMPLE 5. *Solve the following equations:*

$\log_a b = c$

$a^c = b$

(a) $\log_{0.5}(\underbrace{\log(x+120)})=-1$

$\log(x+120) = 0.5^{-1}$

$\log_{10}(x+120) = 2$

$x+120 = 10^2$

$x = 100-120$

$\boxed{x = -20}$

Remark: verify that $x=-20$ satisfies the given equation.

(b) $e^{5+2x} = 4$

$e^b = a$

$\ln a = b$

$5 + 2x = \ln 4$

$2x = \ln 4 - 5$

$x = \frac{1}{2}\ln 4 - \frac{5}{2}$

$x = \ln 4^{\frac{1}{2}} - \frac{5}{2}$

$\boxed{x = \ln 2 - \frac{5}{2}}$

(c) $\log(x-1)+\log(x+1)=\log 15$

$$\log\big((x-1)(x+1)\big)=\log 15$$

$$(x-1)(x+1)=15$$

$$x^2-1=15$$

$$x^2=16$$

$\boxed{x=4}$ or $x=-4$

not a solution because $\log(x-1)$ is undefined at $x=-4$.

EXAMPLE 6. *Find the inverse of the following functions:*

$f(x) = \ln x$
$f^{-1}(x) = e^x$

(a) $f(x) = \ln(x + 12)$

$y = \ln(x+12)$
$x + 12 = e^y$
$x = e^y - 12$ $\longrightarrow$ $y = e^x - 12$

$$\boxed{f^{-1}(x) = e^x - 12}$$

$f(x) = 10^x$

$f^{-1}(x) = \log x$

(b) $f(x) = \dfrac{10^x - 1}{10^x + 1}$

$$y = \frac{10^x - 1}{10^x + 1}$$

$$(10^x + 1)y = 10^x - 1$$

$$10^x y + y = 10^x - 1$$

$$y + 1 = 10^x - 10^x y$$

$$y + 1 = 10^x (1 - y)$$

$$10^x = \frac{y+1}{1-y}$$

$$x = \log \frac{y+1}{1-y}$$

$$y = \log \frac{x+1}{1-x}$$

$$\boxed{f^{-1}(x) = \log \frac{1+x}{1-x}}$$

Change of Base formula:

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

In particular,

$$\log_a x = \frac{\ln x}{\ln a}.$$

EXAMPLE 7. *Using calculator and the change-of-base formula evaluate* $\log_2 15$ *to four decimal places.*

Solution.

$$\log_2 15 = \frac{\ln 15}{\ln 2} \approx 3.9069$$