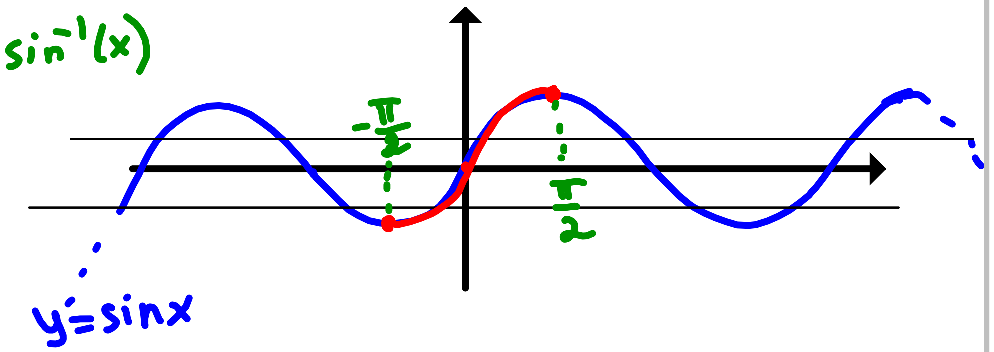
4.6: Inverse trigonometric functions

$\sin^{-1} x \neq \frac{1}{\sin x} = (\sin x)^{-1}$

- **INVERSE SINE:** If $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, then $f(x) = \sin x$ is one-to-one, thus the inverse exists, denoted by $\sin^{-1}(x)$ or $\arcsin x$.

	$y = \sin x$	$y = \arcsin x = \sin^{-1}(x)$
Domain	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$
Range	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$



Cancellation equations:

$f(x) = \sin x$

$f^{-1}(x) = \arcsin x$

$f^{-1}(f(x)) = x$

$$\arcsin(\sin x) = x \quad \text{if} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

and

$f(f^{-1}(x)) = x$

$$\sin(\arcsin x) = x \quad \text{if} \quad -1 \leq x \leq 1.$$

$\sin a = b$ $\qquad$ $\arcsin b = a$

$-\frac{\pi}{2} \le a \le \frac{\pi}{2}$ $\qquad$ $-1 \le b \le 1$

EXAMPLE 1. *Find the exact values of the expression:*

$\sin \boxed{?} = *$

$\sin \boxed{} = 0, -1$

(a) $\sin^{-1} 0$ $= 0$

(b) $\arcsin(-1)$ $= -\frac{\pi}{2}$

(c) $\sin^{-1}(0.5)$ $= \frac{\pi}{6}$

(d) $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ $= -\frac{\pi}{4}$

$\sin(\arcsin x) = x$

$-1 \le x \le 1$

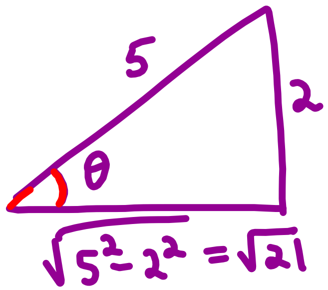
(e) $\sin\left(\arcsin \frac{2}{7}\right) = \frac{2}{7}$ by Cancellation Rule

$-1 \le \frac{2}{7} \le 1$

(f) $\tan\left(\underbrace{\arcsin \frac{2}{5}}_{\theta}\right) = \tan\theta = \frac{2}{\sqrt{21}}$

$\theta = \arcsin \frac{2}{5}$

$\sin\theta = \frac{2}{5}$



(g) $\arcsin\left(\sin\frac{5\pi}{4}\right)$

$\arcsin(\sin x) = x$

$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

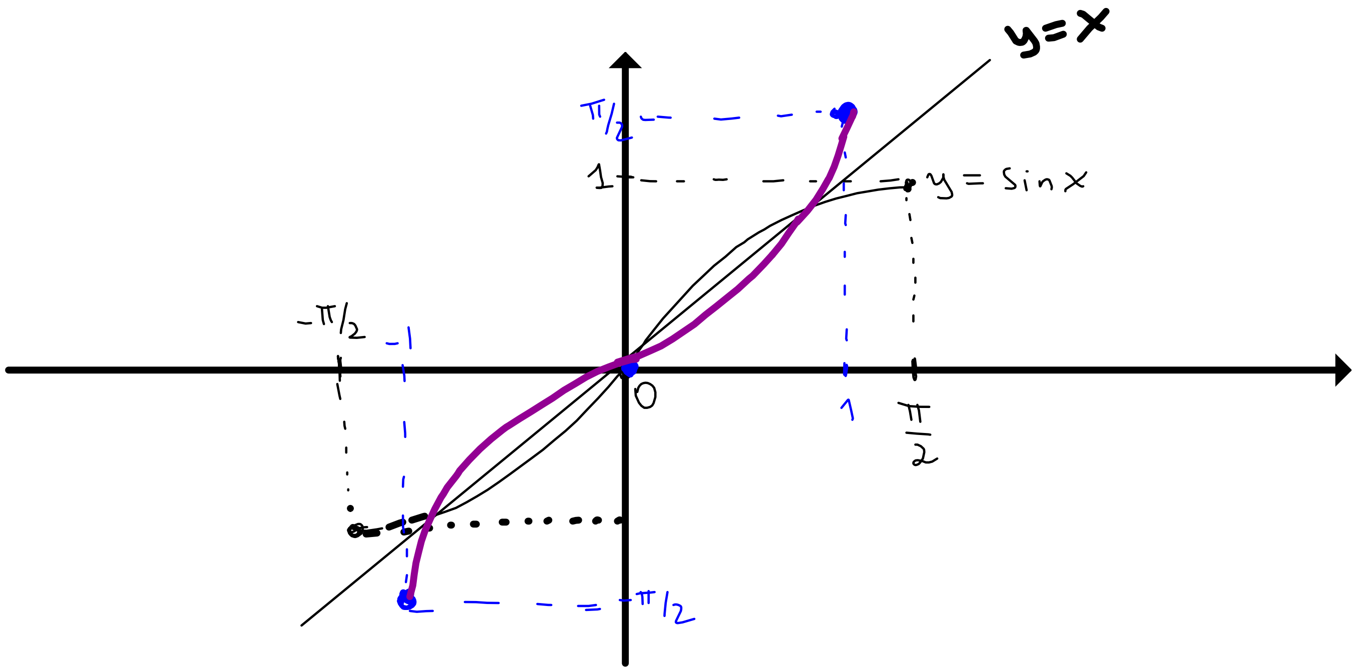
(h) $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$ by Cancellation Rule.

$-\frac{\pi}{2} \le -\frac{\pi}{3} \le \frac{\pi}{2}$

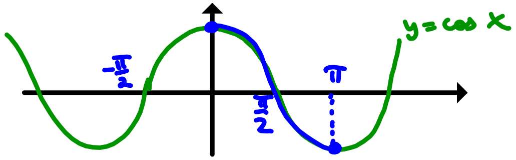
(i) $\arcsin\left(\sin\frac{\pi}{150}\right) = \frac{\pi}{150}$ by Canc. Rule.

$-\frac{\pi}{2} \le \frac{\pi}{150} \le \frac{\pi}{2}$

EXAMPLE 2. *Sketch the graph of* $\arcsin(x)$.



- **INVERSE COSINE:** If $0 \leq x \leq \pi$, then $f(x) = \cos x$ is one-to-one, thus the inverse exists, denoted by $\cos^{-1}(x)$ or $\arccos x$.



	$y = \cos x$	$y = \arccos x$
Domain	$[0, \pi]$	$[-1, 1]$
Range	$[-1, 1]$	$[0, \pi]$

Cancellation equations:

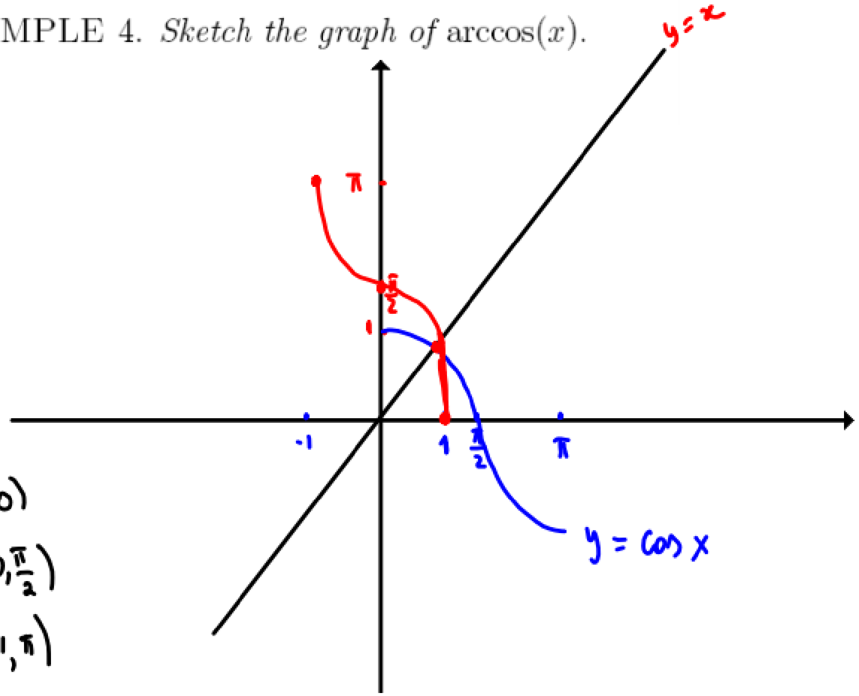
$$\arccos(\cos x) = x \quad \text{if} \quad 0 \leq x \leq \pi$$

and

$$\cos(\arccos x) = x \quad \text{if} \quad -1 \leq x \leq 1.$$

EXAMPLE 3. *Sketch the graph of* $\arccos(x)$.

EXAMPLE 4. *Sketch the graph of* $\arccos(x)$.



$(0,1) \to (1,0)$

$(\frac{\pi}{2}, 0) \to (0, \frac{\pi}{2})$

$(\pi, -1) \to (-1, \pi)$

EXAMPLE 4. *Find the exact values of the expression:*

(a) $\arccos 0 = \frac{\pi}{2}$, $\cos \frac{\pi}{2} = 0$

(b) $\cos^{-1} 1 = 0$, $\cos 0 = 1$

(c) $\arccos(-1) = \pi$, $\cos \pi = -1$

(d) $\arccos 0.5 = \arccos \frac{1}{2} = \frac{\pi}{3}$

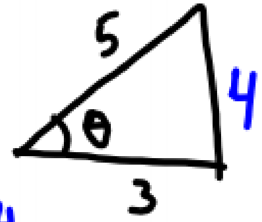
(e) $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$, $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(f) $\sin\left(2\arccos\frac{3}{5}\right) = 2\overbrace{\sin\left(\arccos\frac{3}{5}\right)}^{4/5}\cdot\underbrace{\cos\left(\arccos\frac{3}{5}\right)}_{\frac{3}{5}} =$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\theta = \arccos\frac{3}{5}$

$\cos\theta = \frac{3}{5}$



$\sin\theta = \frac{4}{5} = \sin\left(\arccos\frac{3}{5}\right)$

$= 2\cdot\frac{4}{5}\cdot\frac{3}{5} = \boxed{\frac{2\ldots}{2\ldots}}$

(g) $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

$0 \le \frac{\pi}{6} \le \pi$ Cancel Rule

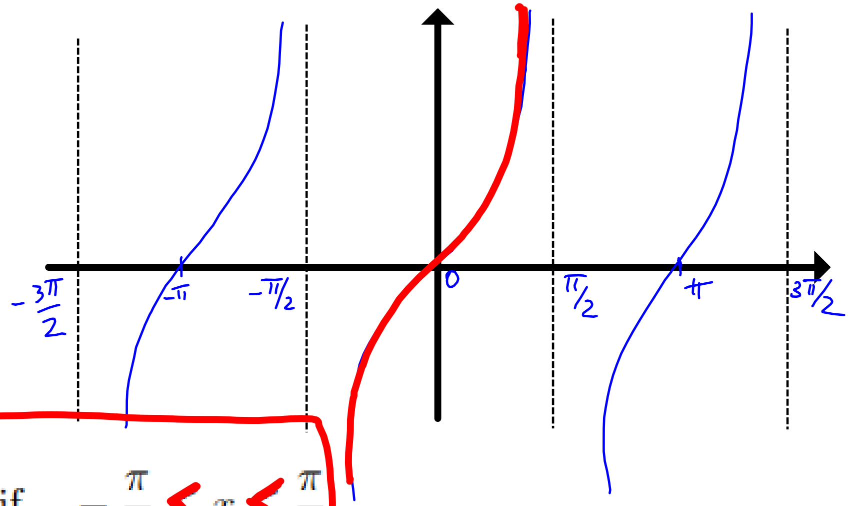
(h) $\arccos\left(\cos\frac{7\pi}{6}\right) = \arccos\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6}$

(i) $\cos(\arccos 2)$ undefined, 2 is not in the domain of arccos

(j) $\arccos\left(\cos\left(-\frac{\pi}{3}\right)\right) = \arccos\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}$

- **INVERSE TANGENT:** If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then $f(x) = \tan x$ is one-to-one, thus the inverse exists, denoted by $\tan^{-1}(x)$ or $\arctan x$.

	$y = \tan x$	$y = \arctan x$
Domain	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(-\infty, \infty)$
Range	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Cancellation equations:

$$\arctan(\tan x) = x \quad \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

and

$$\tan(\arctan x) = x \quad \text{for all} \quad x.$$

EXAMPLE 5. *Find the exact values of the expression:*

$\tan \square = 0$

-1

$\tan \square = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

(a) $\arctan 0 = 0$

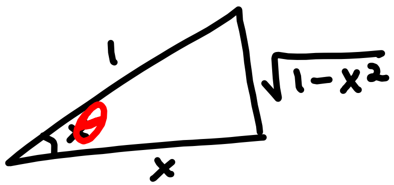
(b) $\arctan(-1) = -\frac{\pi}{4}$

(c) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

(d) $\tan(\underbrace{\arccos x}_{\theta}) = \tan\theta = \frac{\sqrt{1-x^2}}{x}$

$\theta = \arccos x$

$\cos\theta = x = \frac{x}{1}$



(e) $\arctan\left(\tan\frac{5\pi}{4}\right)$ Way 1 $= \arctan(1) = \frac{\pi}{4}$

Way 2 $\arctan\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$ by Cancellation Rule.

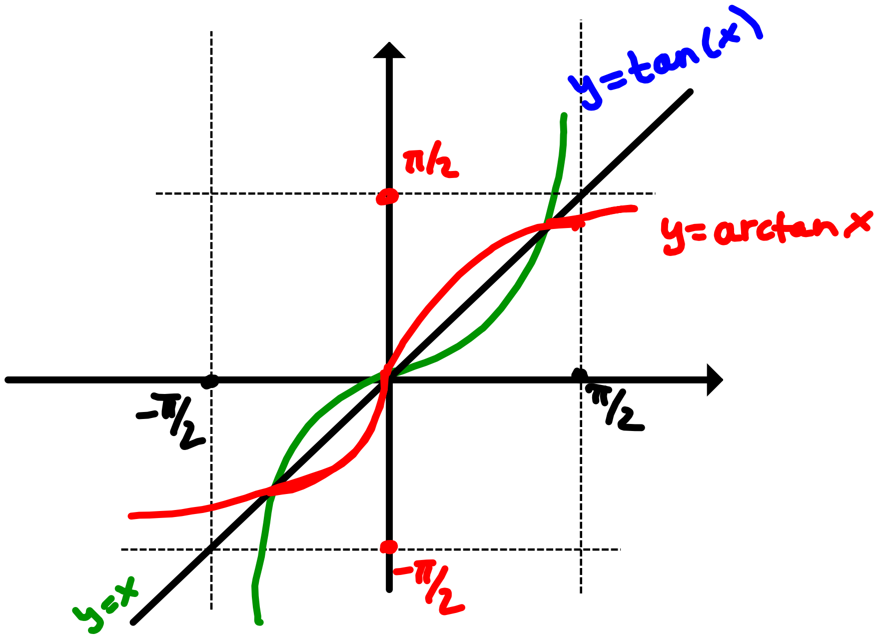
$\tan\left(\frac{5\pi}{4}\right) = \tan\left(\pi + \frac{\pi}{4}\right) = \tan\frac{\pi}{4}$

EXAMPLE 6. *Find the following limits:*

(a) $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$

(b) $\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$

EXAMPLE 7. *Sketch the graph of* $\arctan(x)$.



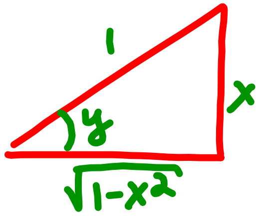
Derivatives of Inverse Trigonometric Functions:

EXAMPLE 8. **(a)** *Find the derivative of* $f(x) = \arcsin x$.

$y = \arcsin x$

$\boxed{\sin y = x}$

$\sin y = \frac{x}{1}$



Use implicit differentiation

$$\frac{d}{dx}\left(\sin y(x)\right) = \frac{d}{dx}(x)$$

$$\cos y = \frac{\sqrt{1-x^2}}{1}$$

$$y' \cos y = 1$$

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{(\arcsin x)' = (\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}}$$

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

$$(\arcsin u)' = \frac{u'}{\sqrt{1-u^2}}$$

(b) *Find* $\dfrac{d}{dx}\left(\dfrac{1}{\underbrace{\arcsin(3x+1)}_{u}}\right) = -\dfrac{\left(\arcsin(\overbrace{3x+1}^{u})\right)'}{\left(\arcsin(3x+1)\right)^2}$

$$= -\frac{\frac{3}{\sqrt{1-(3x+1)^2}}}{\left(\arcsin(3x+1)\right)^2} = -\frac{3}{\sqrt{1-(3x+1)^2}\left(\arcsin(3x+1)\right)^2}$$

TABLE OF DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}$$

EXAMPLE 9. *Find the derivative of* $f(x) = \sin^{-1}(\overbrace{\arctan x}^{u})$

$$f'(x) = \frac{(\arctan x)'}{\sqrt{1-(\arctan x)^2}} = \frac{1}{(1+x^2)\sqrt{1-(\arctan x)^2}}$$

EXAMPLE 10. *Find domain of the following functions:*

(a) $f(x) = \arcsin(\underbrace{4x + 2}_{u})$

$$\text{Dom}(\arcsin u) = [-1, 1] = \{u \mid -1 \le u \le 1\}$$

$$-1 \le 4x+2 \le 1$$

$$-1-2 \le 4x+2-2 \le 1-2$$

$$-3 \le 4x \le -1$$

$$-\frac{3}{4} \le x \le -\frac{1}{4}$$

$$\boxed{\text{Dom}(\arcsin(4x+2)) = \left[-\frac{3}{4}, -\frac{1}{4}\right]}$$

(b) $f(x) = \arctan(\underbrace{4x+2}_{u})$

$$\text{Dom}(\arctan u) = \boxed{(-\infty, \infty) = \text{Dom}(\arctan(4x+2))}$$

$u = 4x+2$

$u-2 = 4x$

$x = \frac{u-2}{4}$

If u is in $\mathbb{R}$, then x is in $\mathbb{R}$