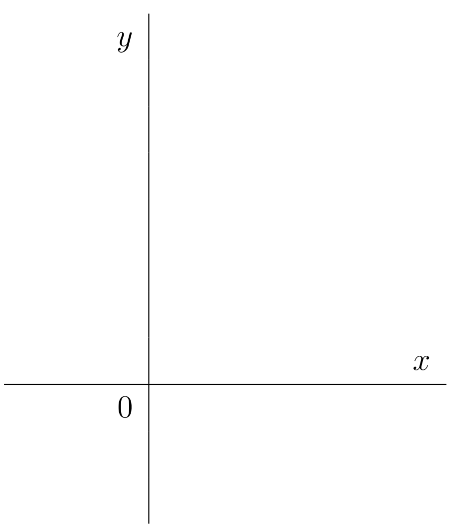
4.8:Indeterminate forms and L'Hospital's Rule

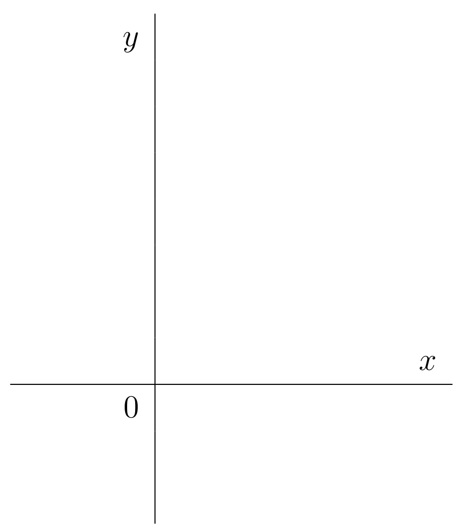
Indeterminate forms: Consider
$$\lim_{x\to a}\frac{f(x)}{g(x)}. \tag{1}$$

- If both $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$, then (1) is called an **indeterminate form of type $\frac{0}{0}$.**
- If both $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$, then (1) is called an **indeterminate form of type $\frac{\infty}{\infty}$.**

EXAMPLES:
$\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\underline{\quad}$, $\lim\limits_{x\to 1}\dfrac{x-x^2}{x^2-1}=\underline{\quad}$, $\lim\limits_{x\to 0}\dfrac{3^x-1}{x^2}=\underline{\quad}$, $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^3}=\underline{\quad}$, $\lim\limits_{x\to\infty}\dfrac{x^2+1}{4x^2-1}=\underline{\quad}$.

L'HOSPITAL'S RULE: Suppose f and g are differentiable and $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ then
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$
if the limit on the right side exists (or is ∞ or $-\infty$).





EXAMPLE 1. *Evaluate each of the following limits:*

(a) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$

(b) $\lim\limits_{x\to\infty}\dfrac{x^2}{e^x}$

(c) $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^3}$

(d) $\lim\limits_{x\to\infty}\dfrac{(\ln x)^5}{x^4}$

Indeterminate form of type $0\cdot\infty$: $\lim\limits_{x\to a} f(x)g(x)$

Write the product fg as a quotient to get an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$:

EXAMPLE 2. *Evaluate each of the following limits:*

(a) $\lim\limits_{x\to 0^+} x^2 \ln x$

(b) $\lim\limits_{x\to -\infty} xe^x$

Indeterminate form of type $\infty - \infty$: $\lim\limits_{x\to a}(f(x) - g(x))$

Try to convert the difference into a quotient to get an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

EXAMPLE 3. *Find:* $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right)$

Indeterminate form of type 0^0**,** ∞^0**,** 1^∞: $\lim\limits_{x\to a} f(x)^{g(x)}$

Write the function as an exponential $0 \cdot \infty$:

It leads to an indeterminate form of type $0 \cdot \infty$.

EXAMPLE 4. *Find the following limits:*

(a) $\lim_{x\to\infty} x^{\frac{1}{x}} =$

(b) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x} =$

(c) $\lim_{x\to 0^+} (\sin x)^{\tan x} =$