4.8:Indeterminate forms and L'Hospital's Rule

Indeterminate forms: Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}. \tag{1}$$

- If both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then (1) is called an **indeterminate form of type** $\frac{0}{0}$.
- If both $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then (1) is called an **indeterminate form of type** $\frac{\infty}{\infty}$.

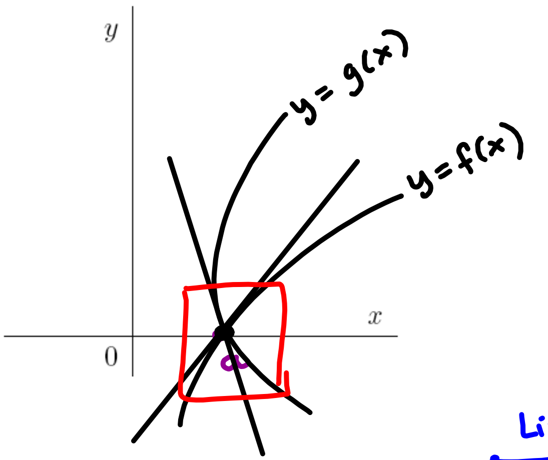
EXAMPLES:

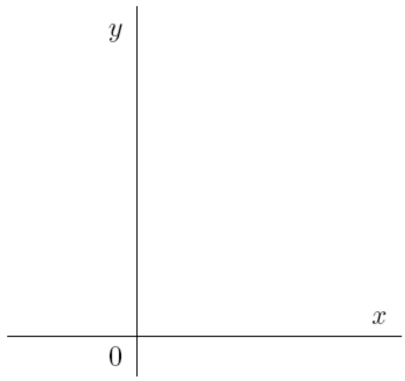
$$\lim_{x\to 0}\frac{\sin x}{x} \approx \frac{0}{0}, \quad \lim_{x\to 1}\frac{x-x^2}{x^2-1} \approx \frac{0}{0}, \quad \lim_{x\to\infty}\frac{\ln x}{x^3} \approx \frac{\infty}{\infty},$$

L'HOSPITAL'S RULE: Suppose f and g are differentiable and $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).





$$f(x) \approx \overbrace{\cancel{f(a)}^{\to 0} + f'(a)(x-a)}^{\text{Linearization}}$$

$$g(x) \approx \cancel{g(a)}^{\to 0} + g'(a)(x-a)$$

near $x=a$

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)}{g'(a)}$$

EXAMPLE 1. *Evaluate each of the following limits:*

(a) $\lim_{x\to 0} \frac{\sin x}{x} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0} \frac{(\sin x)'}{x'} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1.$

(b) $\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to\infty} \frac{(x^2)'}{(e^x)'} = \lim_{x\to\infty} \frac{2x}{e^x} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to\infty} \frac{2}{e^x} = 0$

(c) $\lim_{x\to 0} \frac{\sin x - x}{x^3} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0} \frac{-\sin x}{6x} \overset{\frac{0}{0}}{\underset{H}{=}}$

$= \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x}\right) = \lim_{x\to 0} 1 = 1$$

Indeterminate form of type $0\cdot\infty$: $\lim_{x\to a} f(x)g(x)$

Write the product fg as a quotient to get an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

$$\frac{g}{\frac{1}{f}} \overset{\frac{\infty}{\infty}}{=} f\cdot g \overset{0\cdot\infty}{=} \frac{f}{\frac{1}{g}} \overset{\frac{0}{0}}{=}$$

EXAMPLE 2. *Evaluate each of the following limits:*

(a) $\lim_{x\to 0+} x^2\ln x \overset{0\cdot\infty}{=}$

$$\lim_{x\to 0^+}\frac{x^2}{\left(\frac{1}{\ln x}\right)} \overset{\frac{0}{0}}{=} \ldots$$

OR

$$\lim_{x\to 0^+}\frac{\ln x}{\left(\frac{1}{x^2}\right)} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{2}{x^3}}$$

$$= \lim_{x\to 0^+}\frac{x^{-1}}{-2x^{-3}} = \lim_{x\to 0^+}\frac{1}{-2x^{-2}}$$

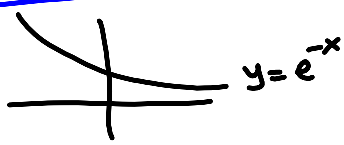
$$= \lim_{x\to 0^+} -\frac{x^2}{2} = 0.$$

(b) $\lim_{x\to -\infty} xe^x \overset{\infty\cdot 0}{=}$

$$\lim_{x\to -\infty}\frac{e^x}{\frac{1}{x}}$$

OR

$$\lim_{x\to -\infty}\frac{x}{e^{-x}} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to -\infty}\frac{1}{-e^{-x}} = 0.$$



(c) $\lim_{x\to\pi/4} (1-\tan x)\sec(2x) \overset{0\cdot\infty}{=} \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\cos(2x)} \overset{\frac{0}{0}}{\underset{H}{=}}$

$$= \lim_{x\to\frac{\pi}{4}} \frac{(1-\tan x)'}{(\cos 2x)'} = \lim_{x\to\frac{\pi}{4}} \frac{-\sec^2 x}{-2\sin(2x)}$$

$$= \frac{\sec^2\frac{\pi}{4}}{2\sin\left(2\,\frac{\pi}{4}\right)} = \frac{(\sqrt{2})^2}{2\cdot 1} = 1$$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\sec\frac{\pi}{4} = \sqrt{2}$

Indeterminate form of type $\infty - \infty$: $\lim_{x\to a}(f(x) - g(x))$

Try to convert the difference into a quotient to get an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

EXAMPLE 3. *Find:* $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$ $\overset{\infty-\infty}{=} \lim_{x\to 1} \frac{x-1-\ln x}{(x-1)\ln x} \overset{\frac{0}{0}}{\underset{H}{=}}$

$$\lim_{x\to 1} \frac{(x-1-\ln x)'}{((x-1)\ln x)'} = \lim_{x\to 1} \frac{1-\frac{1}{x}}{\ln x + (x-1)\frac{1}{x}}$$

$$= \lim_{x\to 1} \frac{x-1}{x\ln x + x - 1} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 1} \frac{(x-1)'}{(x\ln x + x - 1)'}$$

$$= \lim_{x\to 1} \frac{1}{\ln x + \underbrace{x\cdot\frac{1}{x}}_{1} + 1} = \lim_{x\to 1} \frac{1}{\ln x + 2}$$

$$= \frac{1}{\ln 1 + 2} = \frac{1}{0+2} = \boxed{\frac{1}{2}}$$

Indeterminate form of type 0^0, ∞^0, 1^∞: $\lim_{x\to a} f(x)^{g(x)}$

Write the function as an exponential $0\cdot\infty$. It leads to an indeterminate form of type $0\cdot\infty$.

$a = e^{\ln a}$ $\qquad \ln a^b = b \ln a$

EXAMPLE 4. *Find the following limits:*

(a) $\lim_{x\to\infty} x^{\frac{1}{x}} \overset{\infty^0}{=} \lim_{x\to\infty} e^{\ln x^{\frac{1}{x}}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = \lim_{x\to\infty} e^{\frac{\ln x}{x}}$

$= e^{\lim_{x\to\infty}\frac{\ln x}{x}} = e^0 = \boxed{1}$

Find $\lim_{x\to\infty} \frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to\infty} \frac{(\ln x)'}{x'} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

(b) $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \overset{1^\infty}{=} \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$

$\displaystyle = e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)} = e^1 = \boxed{e}$ Final answer.

Find

$\displaystyle \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \overset{\infty\cdot 0}{=} \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} =$

$u=\frac{1}{x}\to 0$ as $x\to\infty$

$\displaystyle = \lim_{u\to 0}\frac{\ln(1+u)}{u} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{u\to 0}\frac{(\ln(1+u))'}{u'} = \lim_{u\to 0}\frac{\frac{1}{1+u}}{1} = \frac{1}{1+0} = 1$

$\displaystyle \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$