1. Determine whether the given integral is convergent or divergent.

(a) $\displaystyle\int_1^{\infty} \frac{4+\cos^4 x}{x}\,\mathrm{d}x$

$$\frac{4+\cos^4 x}{x} \geqslant \frac{4}{x}$$

$0 \le \cos^4 x \le 1$

$\displaystyle\int_1^{\infty} \frac{4}{x}\,dx$ divergent $(p=1)$

By Comp. Theorem the given integral diverges.

(h) $\int_0^\infty \frac{1}{\sqrt{x}+e^{4x}}\,dx$

$a, b > 0 \Rightarrow a+b \geq b$

$\frac{1}{a+b} \leq \frac{1}{b}$

$\cancel{\frac{1}{x}}$ $\frac{1}{e^{4x}}$

$$\frac{1}{\sqrt{x}+e^{4x}} \leq \frac{1}{e^{4x}}$$

$$\int_0^\infty \frac{1}{e^{4x}}\,dx = \int_0^\infty e^{-4x}\,dx = \lim_{t\to\infty} \int_0^t e^{-4x}\,dx$$

$$= \lim_{t\to\infty} \left(-\frac{1}{4}\right) e^{-4x}\Big|_0^t = -\frac{1}{4} \lim_{t\to\infty} (e^{-4t} - e^{0}) = \boxed{\frac{1}{4}}$$

convergent

Thus by Comparison Theorem
the given integral converges.

3. Evaluate $I = \int_0^{2012} \frac{1}{\sqrt{2012 - x}}\, dx$.

improper integral of TYPE II

(incontinuous integrand)

$$I = \lim_{t \to 2012^-} \int_0^t \frac{dx}{\sqrt{2012-x}}$$

$u = 2012 - x$

$du = -dx$

$\int \frac{dx}{\sqrt{2012-x}} = -\int u^{-\frac{1}{2}} du$

$= -2\sqrt{u} = -2\sqrt{2012-x}$

$$= \lim_{t \to 2012^-} -2\sqrt{2012-x}\,\Big|_0^t$$

$$= -2\left[\sqrt{2012-2012} - \sqrt{2012-0}\right] = \boxed{2\sqrt{2012}}$$

3. Set up, *but don't evaluate* the integral for the length of the curve $x = 2t^2, \quad y = t^3, \quad 0 \le t \le 1$.

$$L = \int ds = \int_0^1 \sqrt{(x')^2 + (y')^2}\, dt = \int_0^1 \sqrt{(4t)^2 + (3t^2)^2}\, dt$$

$$\boxed{L = \int_0^1 \sqrt{16t^2 + 9t^4}\, dt}$$

4. Find length of the curve $y = \frac{1}{\pi}\ln(\sec(\pi x))$ from the point $(0,0)$ to the point $(\frac{1}{6}, \ln\frac{2}{\sqrt{3}})$.

$$L = \int ds = \int_0^{1/6} \sqrt{1+(y')^2}\, dx$$

$$y' = \frac{1}{\pi}\left(\ln\left(\underbrace{\sec(\pi x)}\right)\right)' = \frac{1}{\cancel{\pi}} \frac{1}{\cancel{\sec \pi x}} \cdot \cancel{\sec(\pi x)} \tan(\pi x) \cdot \cancel{\pi}$$

$$y' = \tan(\pi x) \quad \Rightarrow 1 + (y')^2 = 1 + \tan^2(\pi x) = \sec^2(\pi x)$$

$$L = \int_0^{1/6} \sqrt{\sec^2(\pi x)}\, dx = \int_0^{1/6} |\sec(\pi x)|\, dx = \int_0^{1/6} \sec(\pi x)\, dx$$

$$L = \frac{1}{\pi} \ln|\sec(\pi x) + \tan(\pi x)| \Big|_0^{1/6}$$

$$L = \frac{1}{\pi}\left[\ln\left|\sec\frac{\pi}{6} + \tan\frac{\pi}{6}\right| - \underbrace{\ln|\sec 0 + \tan 0|}_{\ln 1 = 0}\right]$$

$$L = \frac{1}{\pi}\ln\left|\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{3}\right|$$

5. Write out the form of the partial fraction decomposition (do not try to solve)

$$\frac{20x^3+12x^2+x}{\underbrace{(x^3-x)}_{\substack{x(x^2-1)\\ x(x-1)(x+1)}}\underbrace{(x^3+2x^2-3x)}_{\substack{x(x^2+2x-3)\\ x(x+3)(x-1)}}\underbrace{(x^2+x+1)}\underbrace{(x^2+9)^2}}$$

$$\frac{\cancel{x}(20x^2+12x+1)}{\underset{\text{L.}}{x^{\cancel{2}}}\,\underset{\text{L.R.}}{(x-1)^2}\,\underset{\text{L.}}{(x+1)}\,\underset{\text{L.}}{(x+3)}\,\underset{\text{Q.P.}}{(x^2+x+1)}\,\underset{\text{Q.P.R.}}{(x^2+9)^2}} = \frac{A}{x}+\frac{B_1}{x-1}+\frac{B_2}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{x+3}$$

$$+\frac{Ex+F}{x^2+x+1}+\frac{G_1x+H_1}{x^2+9}+\frac{G_2x+H_2}{(x^2+9)^2}$$

6. Evaluate the integral $\int \frac{5x^2+x+12}{x^3+4x}\,dx = \int \frac{5x^2+x+12}{x(x^2+4)}\,dx$

deg(num) < deg(denom.)

$$\frac{5x^2+x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$5x^2+x+12 = A(x^2+4) + (Bx+C)x$$

$x=0$ $\quad 12 = 4A \Rightarrow \boxed{A=3}$

$x^2:$ $\quad 5 = A+B \Rightarrow 5 = 3+B \Rightarrow \boxed{B=2}$

$x:$ $\quad \boxed{1=C}$

$$\int = \int \frac{A}{x} + \frac{Bx+C}{x^2+4}\,dx = \int \frac{3}{x} + \frac{2x}{x^2+4} + \frac{1}{x^2+4}\,dx$$

$$= 3\ln|x| + \ln(x^2+4) + \frac{1}{2}\arctan\frac{x}{2} + C$$

7. A tank contains 250 liters of pure water. Brine that contains 0.01 kg of salt per liter enters the tank at a rate of 20 liters per minute. The solution is kept mixed and drains from the tank at a rate of 20 liters per minute. How much salt is in the tank after t minutes?

Let $y = y(t)$ be the amount of salt in the tank at time t. Since we are starting with pure water, $y(0) = 0$. Now,

$$\frac{dy}{dt} = \left(0.01\frac{kg}{L}\right)\left(20\frac{L}{m}\right) - \left(\frac{y}{250}\frac{kg}{L}\right)\left(20\frac{L}{m}\right).$$

Thus $\frac{dy}{dt} = 0.2\frac{kg}{m} - \frac{2}{25}y\frac{kg}{m}$. We can solve this equation as linear or separable. I will solve it as separable: $\frac{dy}{dt} = \frac{5-2y}{25}$, thus $\frac{dy}{5-2y} = \frac{1}{25}dt$. Integrating both sides yields

$-\frac{1}{2}\ln(5-2y) = \frac{1}{25}t + C$. Since $y(0) = 0$, this gives $C = -\frac{1}{2}\ln(5)$. $-\frac{1}{2}\ln(5-2y) = \frac{1}{25}t - \frac{1}{2}\ln(5)$. Thus $\ln(5-2y) = -\frac{2}{25}t + \ln(5)$. Solve for y:

$$5 - 2y = e^{-2t/25+\ln 5} = 5e^{-2t/25}.$$

Hence $y = \dfrac{5-5e^{-2t/25}}{2}$ kg of salt.

8. What is the smallest value of n so that the approximation T_n (the trapezoidal rule with n subintervals) to the integral $\int_1^3 \ln x \, dx$ is accurate to within $\frac{1}{2400}$?

$|E_T| \leq \dfrac{K(b-a)^3}{12n^2}$, where $a = 1$, $b = 3$, and $K =$

$\max|f''(x)|$ for $1 \leq x \leq 3 = \max|\dfrac{-1}{x^2}|$ for $1 \leq x \leq 3$,

hence $K = 1$. This yields $|E_T| \leq \dfrac{1(2)^3}{12n^2} \leq \dfrac{1}{2400}$.

This yields $1600 \leq n^2$, hence n must be atleast 40.