1. Find the general solution of the differential equation $ty' + 3y = \cos t$, $t > 0$, and determine how the solutions behave as $t \to +\infty$.

$ty' + 3y = \cos t$, $t > 0$

Use the method of integrating factor

$$y' + \frac{3}{t} y = \frac{\cos t}{t}$$

$$\mu y' + \frac{3}{t}\mu y = \frac{\cos t}{t}\mu$$

Find μ s.t. $\mu' = \frac{3}{t}\mu \Rightarrow$ we can take $\mu = e^{\int \frac{3}{t}\,dt} = e^{3\ln t} = t^3$

$$\underbrace{\mu y' + \mu' y}_{} = (\mu y)' = \frac{\cos t}{t} \cdot t^3 \Rightarrow$$

$$t^3 y = \int t^2 \cos t = t^2 \sin t + 2t\cos t - 2\sin t + C$$

Calculation of the integral

$$\int t^2 \cos t\,dt \underset{\text{by part}}{=} t^2 \sin t - 2\int t \sin t\,dt =$$

$$= t^2 \sin t - \left(-2t\cos t + 2\int \cos t\,dt\right) =$$

$$= t^2 \sin t + 2t\cos t - 2\sin t$$

$$\Downarrow$$

$$\boxed{y(t) = \frac{\sin t}{t} + \frac{2\cos t}{t^2} - \frac{2\sin t}{t^3} + \frac{C}{t^3}}$$

$$\boxed{\lim_{t \to +\infty} y(t) = 0}$$

2. Solve the initial value problem $y' - 5y = te^{4t}, \quad y(0) = a$, where a is an arbitrary real constant.

$y' - 5y = te^{4t}$

Use the method of integrating factor

$\mu y' \underbrace{- 5\mu y}_{\mu'} = te^{4t}\mu$

$\mu' = -5\mu \Rightarrow \mu = e^{-5t}$

$\int te^{-t} = -te^{-t} + \int e^{-t} = -te^{-t} - e^{-t}$

$\underbrace{(\mu y)'}_{\mu y' + \mu' y} = te^{4t}e^{-5t} = te^{-t} \Rightarrow$

$e^{-5t}y = -te^{-t} - e^{-t} + C \Rightarrow$

$y = -te^{4t} - e^{4t} + Ce^{5t}$

$y(0) = a \Rightarrow a = -1 + C \Rightarrow C = a+1$

$\boxed{y(t) = -te^{4t} - e^{4t} + (a+1)e^{5t}}$

$= -e^{4t}(t+1) + (a+1)e^{5t}$

3. Given a series whose partial sums are given by $s_n = (7n+3)/(n+7)$, find the general term a_n of the series and determine if the series converges or diverges. If it converges, find the sum.

$$S_n = \frac{7n+3}{n+7}$$

$$S_n = S_{n-1} + a_n \Rightarrow a_n = S_n - S_{n-1}$$

$$a_n = \frac{7n+3}{n+7} - \frac{7(n-1)+3}{n-1+7}$$

$$\boxed{a_n = \frac{7n+3}{n+7} - \frac{7n-4}{n+6}}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{7n+3}{n+7} = \frac{7}{1} = 7$$

$\Rightarrow$ The series converges and its sum is 7.

4. Find the sum of the following series or show they are divergent:

(a) $$\sum_{n=1}^{\infty} \frac{7+5^n}{10^n} = \sum_{n=1}^{\infty} 7\cdot\left(\frac{1}{10}\right)^n + \left(\frac{5}{10}\right)^n = \sum_{n=1}^{\infty} \underbrace{7\cdot\left(\frac{1}{10}\right)^n}_{\text{geometric series}} + \underbrace{\left(\frac{1}{2}\right)^n}$$

$$= \sum_{n=1}^{\infty} \underbrace{7\cdot\frac{1}{10}}_{a}\cdot\left(\frac{1}{10}\right)^{n-1} + \underbrace{\frac{1}{2}}_{a}\cdot\left(\frac{1}{2}\right)^{n-1}$$

$r = \frac{1}{10}$ $\qquad$ $r = \frac{1}{2}$

In both cases $-1 < r < 1 \Rightarrow$ the geom. series converges

Use formula $\frac{a}{1-r}$ to find sum

$$= \frac{7/10}{1-1/10} + \frac{1/2}{1-\frac{1}{2}} = \frac{7}{9} + 1 = \boxed{\frac{16}{9}}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+3)} = \sum_{n=1}^{\infty} \overbrace{\frac{4}{n+1} - \frac{4}{n+3}}^{a_n}$

Telescoping series

$$\frac{8}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3}$$

$$8 = A(n+3) + B(n+1)$$

$n=-1 \Rightarrow 8 = 2A \Rightarrow A = 4$

$n=-3 \Rightarrow 8 = -2B \Rightarrow B = -4$

$$\boxed{\frac{4}{n+1} - \frac{4}{n+3} = a_n}$$

$$S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n = \underbrace{\boxed{\tfrac{4}{2}} - \cancel{\tfrac{4}{4}}}_{n=1} + \underbrace{\boxed{\tfrac{4}{3}} - \cancel{\tfrac{4}{5}}}_{n=2} + \underbrace{\cancel{\tfrac{4}{4}} - \cancel{\tfrac{4}{6}}}_{n=3} + \ldots$$

$$+ \ldots + \ldots \underbrace{\cancel{\tfrac{4}{n}} \boxed{- \tfrac{4}{n+2}}}_{n \to n-1} + \underbrace{\cancel{\tfrac{4}{n+1}} \boxed{- \tfrac{4}{n+3}}}_{n}$$

$$S_n = \underbrace{\frac{4}{2} + \frac{4}{3}}_{2 + \frac{4}{3} = \frac{10}{3}} - \underset{\downarrow \atop 0}{\frac{4}{n+2}} - \underset{\downarrow\, n\to\infty \atop 0}{\frac{4}{n+3}}$$

$$\lim_{n\to\infty} S_n = \boxed{\frac{10}{3}}$$ sum of the series.

The series converges.

5. Write the repeating decimal $0.\overline{27}$ as a fraction.

$0.\overline{27} = 0.27272727\ldots$

$$\begin{aligned} &= 0.27 \\ &+ 0.0027 \\ &+ 0.000027 \\ &\ 0.00000027 \\ &+ \ldots \end{aligned}$$

$$\begin{aligned} &\ 0.27 \\ &+ 0.27 \cdot 10^{-2} \\ = &+ 0.27 \cdot 10^{-4} \\ &+ 0.27 \cdot 10^{-6} \\ &\ldots \end{aligned}$$

$\times 10^{-2}$ $\times 10^{-2}$ $\times 10^{-2}$

sum of geometric series

with $a = 0.27$, $r = 10^{-2} = 0.01$

$$0.\overline{27} = \frac{a}{1-r} = \frac{0.27}{1-0.01} = \frac{0.27}{0.99} = \boxed{\frac{27}{99}} = \boxed{\frac{3}{11}}$$

6. Use the test for Divergence to determine whether the series diverges.

(a) $\sum_{n=1}^{\infty} \frac{n^5}{3(n^4+3)(n+1)}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^5}{3(n^4+3)(n+1)} = \frac{1}{3}$$

$\Rightarrow$ the series diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$

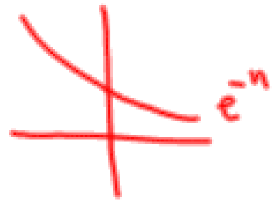
$$\lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0 \Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n\sqrt{n}} = 0$$

no conclusion at the moment

because DT fails

Use another test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6 - e^{-n}}$

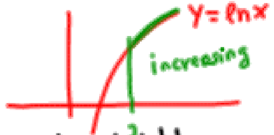


$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{6-\underbrace{e^{-n}}_{\to 0}} = \frac{1}{6}$$

The series diverges

7. Determine if the sequence $\{a_n\}_{n=2}^{\infty}$ is decreasing and bounded:

(a) $a_n = \ln n$ $\qquad f(x) = \ln x$



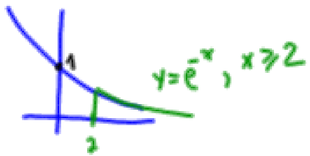
$\{a_n\}$ is not decreasing and unbdd

(b) $a_n = \cos n^2$ $\qquad$ not decreasing

oscillate $\qquad$ bounded, because $-1 \le \cos n^2 \le 1$

(c) $a_n = e^{-n}$

$f(x) = e^{-x}$



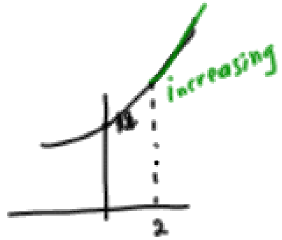
decreasing

bounded

$\left(0 < e^{-x} < 1 \text{ for } x \ge 2 \right)$

(d) $a_n = e^n + 11$

$f(x) = e^x + 11$



not decreasing

not bdd

(e) $a_n = 1 - \frac{1}{n^2}$

$0 < a_n < 1$ $\quad$ bounded

$f(x) = 1 - \frac{1}{x^2} \Rightarrow f'(x) = \frac{1}{x} > 0$ for $x \ge 2$

increasing

not decreasing

8. Determine if the sequence converges or diverges. If converges, find its limit.

(a) $\left\{ \dfrac{2012 + (-1)^n}{n^{2012}} \right\}_{n=1}^{\infty}$

$$\frac{2012-1}{n^{2012}} \le \frac{2012+(-1)^n}{n^{2012}} \le \frac{2012+1}{n^{2012}}$$

$$\frac{2012-1}{n^{2012}} \xrightarrow{n\to\infty} 0 \qquad \frac{2012+(-1)^n}{n^{2012}} \xrightarrow{\text{squeeze Theorem}} 0 \qquad \frac{2012+1}{n^{2012}} \xrightarrow{n\to\infty} 0$$

The sequence converges to zero.

(b) $\left\{ \sqrt{\dfrac{7n + 6n^3 + n^2}{(n+3)(n^2+8)}} \right\}_{n=4}^{\infty} \longrightarrow \boxed{\sqrt{6}}$

$$\parallel$$

$$\sqrt{\frac{6n^3 + \cdots}{n^3 + \cdots}}$$ rational function

9. Assuming that the sequence defined recursively by $a_1 = 1$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{16}{a_n}\right)$ is convergent, find its limit.

Denote $\lim_{n\to\infty} a_n = L > 0$

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = L$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{16}{a_n}\right) = \frac{1}{2}\left(L + \frac{16}{L}\right) = L$$

$$L + \frac{16}{L} = 2L$$

$$\frac{L^2+16}{L} = 2L \Rightarrow L^2+16 = 2L^2$$

$$L^2 = 16$$

$$L = \pm 4$$

$$\boxed{L = 4}$$

10. For what values of x the series $\sum_{n=0}^{\infty}(4x-3)^{n+3}$ converges? What is the sum of the series?

Note that the series is geometric

$$\sum_{n=0}^{\infty}(4x-3)^{n+3} = \sum_{n=0}^{\infty}\underbrace{(4x-3)^3}_{a}\,\underbrace{(4x-3)^n}_{r^n}$$

$r = 4x-3$

Geom. series converges $\Leftrightarrow$ $|r|<1$

$$-1 < r < 1$$
$$-1 < 4x-3 < 1$$
$$-1+3 < 4x < 1+3$$
$$2 < 4x < 4$$
$$\frac{1}{2} < x < 1$$

For $\boxed{\frac{1}{2} < x < 1}$ we have

$$\sum_{n=0}^{\infty}(4x-3)^{n+3} = \frac{a}{1-r} = \frac{(4x-3)^3}{1-(4x-3)} = \boxed{\frac{(4x-3)^3}{4-4x}}$$

11. Compute $S = \sum_{n=1}^{\infty} \overbrace{(\underbrace{e^{1/n}}_{b_n} - \underbrace{e^{1/(n+1)}}_{b_{n+1}})}^{a_n}$.

$$S_n = a_1 + a_2 + \dots + a_{n-1} + a_n = b_1 - \cancel{b_2} + \cancel{b_2} - \cancel{b_3} + \dots + \cancel{b_{n-1}} - \cancel{b_n} + \cancel{b_n} - b_{n+1}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (b_1 - b_{n+1}) = \lim_{n\to\infty} \left(e - e^{\frac{1}{n+1}}\right)$$

$$= \boxed{e-1}$$

12. Which of the following series converges absolutely?

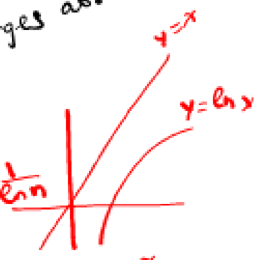
NO (a) $\sum_{n=1}^{\infty}(-1)^{n+5}$ — $\lim_{n\to\infty}(-1)^{n+5}$ DNE $\Rightarrow$ the series Diverges by DT

YES (b) $\sum_{n=1}^{\infty}\frac{\sin(\pi^3 n^2)}{n^2\sqrt{n}}$; $\left|\frac{\sin \pi^3 n^2}{n^2\sqrt{n}}\right| < \frac{1}{n^{5/2}}$) by Comp. Test the series of abs. values converges

$\sum \frac{1}{n^{5/2}}$ convergent $\Rightarrow$ the given series converges absolutely

NO (c) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[4]{n}}$ b/c $\sum \frac{1}{\sqrt[4]{n}}$ diverges, $p=\frac{1}{4}<1$

NO (d) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln n}$ — $\frac{1}{\ln n} > \frac{1}{n}$, $\sum \frac{1}{n}$ divergent $\Big\}$ by Comp. Test the series $\sum \frac{1}{\ln n}$ diverges



$0 < \ln x < x$, $x \geq 2$, $\frac{1}{\ln x} > \frac{1}{x}$

YES (e) $\sum_{n=1}^{\infty}\frac{n^n}{(n!)^2}$

Ratio Test

$$l=\lim_{n\to\infty}|a_{n+1}|\cdot\frac{1}{|a_n|}=\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!^2}\cdot\frac{n!^2}{n^n}$$

$$=\lim_{n\to\infty}\frac{(n+1)^n\cdot\cancel{(n+1)}}{\cancel{n!^2}(n+1)^{\cancel{2}}}\cdot\frac{\cancel{n!^2}}{n^n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{1}{n+1}=0<1$$

the series conv. absolutely

NO (f) $\sum_{n=1}^{\infty}\frac{5^n}{\ln(n+1)}$

$$\lim_{n\to\infty}\frac{5^n}{\ln(n+1)}=\frac{\infty}{\infty}\ \text{L'Hospital's}$$

$$=\lim_{n\to\infty}\frac{5^n\ln 5}{\frac{1}{n+1}}$$

$$=\ln 5\lim_{n\to\infty}5^n(n+1)=\infty$$

The series diverges by DT.

$R=4$

13. Suppose that the power series $\sum_{n=1}^{\infty} c_n(x-4)^n$ has the radius of convergence 4. Consider the following pair of series:

$$\boxed{(I) \quad \sum_{n=1}^{\infty} c_n 5^n} \qquad (II) \quad \sum_{n=1}^{\infty} c_n 3^n.$$

$x-4=5 \Rightarrow x=9$
divergent

$x-4=3 \Rightarrow x=7$
converges

Which of the following statements is true?

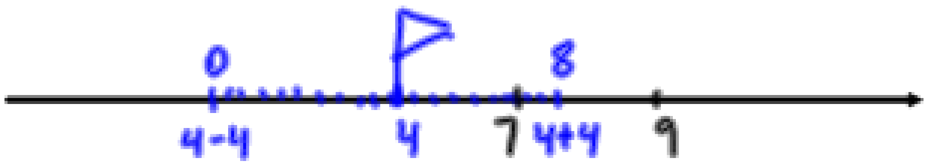
(a) (I) is convergent, (II)is divergent

(b) Neither series is convergent

(c) Bith series are convergent

(d) (I) is divergent, (II) is convergent

(e) no conclusion can be drawn about either series.



14. Show that the series $\sum^{\infty} \frac{\ln n}{n^2}$ converges. Then find un upper bound on the error in using s_{10} to approximate

Show that $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges

Let's see if $f(x) = \frac{\ln x}{x^2}$ is easy to integrate

$\int \frac{\ln x}{x^2}\,dx = \int x^{-2} \ln x \, dx$

$u = \ln x$; $dv = x^{-2}dx$

$du = \frac{dx}{x}$; $v = -x^{-1}$

$= -x^{-1}\ln x - \int -x^{-1}\frac{dx}{x} = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x}$

$= -\left[\frac{\ln x}{x} + \frac{1}{x}\right]$

$f(x) = \frac{\ln x}{x^2}$ is positive, continuous on $[2, \infty)$

Is $f(x)$ decreasing? $f'(x) = \frac{\frac{1}{x}\cdot x^2 - 2x\ln x}{x^4}$

$= \frac{x(1-2\ln x)}{x^4} < 0$ decreasing.

By Integral Test, the series conv./diverges together with $\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_2^t \frac{\ln x}{x^2} =$

$= \lim_{t\to\infty} -\left[\frac{\ln x}{x} + \frac{1}{x}\right]\Big|_2^t$

$= -\lim_{t\to\infty} \underbrace{\frac{\ln t}{t} + \frac{1}{t}}_{\to 0 \text{ (use L'Hospitals Rule)}} - \frac{\ln 2}{2} - \frac{1}{2}$

$= \frac{\ln 2}{2} + \frac{1}{2}$

The series converges

Find un upper bound for R_{10}

$\sum_{n=2}^{\infty} \frac{\ln n}{n^2} \simeq S_{10}$

By Integral Test Remainder estimate:

$R_n \leq \int_n^{\infty} f(x)\,dx$

$R_{10} \leq \int_{10}^{\infty} \frac{\ln x}{x^2}\,dx = \frac{\ln 10}{10} + \frac{1}{10}$

$\simeq 0.33$

15. If we represent $\frac{x^2}{4+9x^2}$ as a power series centered at $a = 0$, what is the associated radius of convergence?

$$f(x) = \frac{x^2}{4+9x^2} = x^2 \cdot \frac{1}{4+9x^2}$$

$$= x^2 \cdot \frac{1}{4} \cdot \frac{1}{1+\frac{9x^2}{4}} = \frac{x^2}{4} \cdot \underbrace{\frac{1}{1-\left(-\frac{9x^2}{4}\right)}}_{\text{sum of geometric series with common ratio}}$$

$$|r| = \left|-\frac{9x^2}{4}\right| < 1$$

$$-1 < \frac{9x^2}{4} < 1 \qquad \sqrt{x^2} = |x|$$

$$-\frac{4}{9} < x^2 < \frac{4}{9}$$

$$|x| < \frac{2}{3} \Rightarrow \boxed{R = \frac{2}{3}}$$

16. Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-2)^n(3x-1)^n}{n}$.

$$|a_n| = \frac{2^n |3x-1|^n}{n}$$

$$|a_{n+1}| = \frac{2^{n+1} |3x-1|^{n+1}}{n+1}$$

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|}$$

$$= \lim_{n\to\infty} \frac{2^{\cancel{n+1}} |3x-1|^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{2^n} \cancel{|3x-1|^n}} =$$

$$2|3x-1| \lim_{n\to\infty} \frac{n}{n+1} = \underbrace{2|3x-1| < 1}$$

$$|3x-1| < \frac{1}{2}$$

$$\underset{\text{center}}{|x-\tfrac{1}{3}|} < \underset{\text{radius}}{\tfrac{1}{6}} \qquad \boxed{R = \frac{1}{6}}$$

Interval of converges
(L = 1) Test end points

$$2|3x-1| = 1$$

$$3x-1 = \pm\frac{1}{2}$$

$3x-1 = -\frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{(-2)^n \cdot (-\frac{1}{2})^n}{n} =$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ divergent}$$

$3x-1 = \frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{(-2)^n (\frac{1}{2})^n}{n} =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges by AST

$$-\frac{1}{2} < 3x-1 \le \frac{1}{2}$$

$$-\frac{1}{2}+1 < 3x-1+1 \le \frac{1}{2}+1$$

$$\frac{1}{2} < 3x \le \frac{3}{2}$$

$$\boxed{\frac{1}{6} < x \le \frac{1}{2}} \text{ OR } \left(\frac{1}{6}, \frac{1}{2}\right]$$

17. Which of the following statements is TRUE?

F (a) If $a_n > 0$ for $n \geq 1$ and $\sum_{n=1}^{\infty} (-1)^n a_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

T (b) If $a_n > 0$ for $n \geq 1$ and $\underline{\sum_{n=1}^{\infty} a_n \text{ converges}}$ then $\underline{\sum_{n=1}^{\infty} (-1)^n a_n \text{ converges}}$.

F (c) If $\underline{\lim_{n\to\infty} a_n = 0}$ then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges. (a_n is not positive or $\{a_n\}$ is not decreasing)

F (d) If $a_n > 0$ for $n \geq 1$ and $\underbrace{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{e}{2}}_{\text{Ratio Test}}$ then $\sum_{n=1}^{\infty} a_n$ converges. $e \approx 2.7$, $\frac{e}{2} > 1 \Rightarrow \sum a_n$ diverges

(a) $a_n = \frac{1}{n} > 0$

$\sum \frac{(-1)^n}{n}$ converges

but $\sum \frac{1}{n}$ diverges

(b) $\sum a_n$ conv. $\Rightarrow$ $\sum (-1)^n a_n$ converges absolutely $\Rightarrow$ converges.

18. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 3^n}$ converges to s. Use the Alternating Series Theorem to estimate $|s - s_6|$.

$$s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 3^n}$$

$$|s - s_6| = |R_6|$$

We know $\sum_{n=1}^{\infty} (-1)^n b_n$

$$|R_n| \leq b_{n+1}$$

In our case $b_n = \frac{1}{n^2 3^n}$

$$|R_6| \leq b_{6+1} = b_7 = \boxed{\frac{1}{7^2 \cdot 3^7}}$$