Math 172 Exam 1
KEY POINTS (sections 6.5-6.6, 7.1-7.5, 8.1-8.3.)

Note: It is also expected that you demonstrate that you have learned the appropriate definitions, properties and theorems with proofs.

6.5: The substitution rule

If $u = g(x)$ is a differentiable function, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

You should make sure that the old variable x has disappeared from the integral.

6.6: The Logarithm Defined As An Integral

- The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t, \quad x > 0. \tag{1}$$

7.1:Area Between Curves

CASE I $A = \int_a^b \left(\text{upper function}\right) - \left(\text{lower function}\right)\mathrm{d}x$

CASE II $A = \int_c^d \left(\text{right function}\right) - \left(\text{left function}\right)\mathrm{d}y$

- In some cases the limits of integration can be determined as the intersection points of two curves.
- Sketch of a graph of the region is highly recommended.
- The area between two curves will always be positive.

7.2: VOLUME

Cross sections are perpendicular to the axis of rotating. $A(x)$ is crossectional area.

- $V = \int_a^b A(x)\,\mathrm{d}x$ where $A(x)$ is the area of a moving cross-section obtained by slicing through x **perpendicular** to the x-axis.
- $V = \int_c^d A(y)\,\mathrm{d}x$ where $A(y)$ is the area of a moving cross-section obtained by slicing through y **perpendicular** to the y-axis.

7.3: Volumes by Cylindrical Shells

- Area of cross sections: $A = 2\pi(\text{radius})(\text{height})$.
  - For rotation about a *vertical* axis we use $V = \int_a^b A(x)\,\mathrm{d}x = 2\pi\int_a^b r(x)h(x)\,\mathrm{d}x$.
    Note $h(x)$=(Top−Bottom).
  - For rotation about a *horizontal* axis we use $V = \int_c^d A(y)\,\mathrm{d}y = 2\pi\int_c^d r(y)h(y)\,\mathrm{d}y$.
    Note $h(y)$=(Right−Left)

  Note: *Exactly opposite of washer method.*
- For the limits of integration we take the only range of x or y covering one side of the solid (not the complete range).
- In many problems both washer method and shell method can be used. In some problems both take about the same amount of work, but sometimes one is definitely easier than the other.

7.4: Work

- Work W done in moving an object under
  - a constant force F a distance d is $W = Fd$.
  - under a variable force $F(x)$ (x is displacement) from $x = a$ to $x = b$ is $W = \int_a^b F(x)\,\mathrm{d}x$
- Hooke's law: the force required to stretch a spring x units **beyond** its natural length is $F = kx$.
- "Water Pumping" problems require partition along vertical axis (called y here) and work is given by

$$W = \rho g \int_a^b (Area)(distance)\,\mathrm{d}y$$

  where
  - $y = 0$ can be top, bottom, or center (circular ends);
  - ρg is weight density of liquid;
  - (Area) is cross sectional area of horizontal slice;
  - (distance) is the distance the slice travels to reach the top.
- "Rope/cable pulling" problems require partition along vertical axis (called y here). If $y = 0$ is at the top of rope/cable then a work required to pull s feet (or s meter) of a rope/cable that weighs ω lb/ft (or ω N/m) is given by

$$W = \int_0^s \omega y\,\mathrm{d}y$$

7.5: Average Value of a Function

- The **average value of a function** $y = f(x)$ over the interval $[a, b]$:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

- MEAN VALUE THEOREM FOR INTEGRALS: *If f is continuous on $[a, b]$, then there exists a number c on $[a, b]$ s.t.*

$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a).$$

8.1: Integration By Parts

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

$$\int_a^b u\,\mathrm{d}v = uv\big|_a^b - \int_a^b v\,\mathrm{d}u \quad \text{where} \quad uv\big|_a^b = u(b)v(b) - u(a)v(a).$$

To choose u use the LIPET rule, in order of preference for u:

L Logarithmic functions

I Inverse trigonometric functions

P Polynomial functions

E Exponential functions

T Trigonometric functions

8.2: Trigonometric Integrals

- How to evaluate $\int \sin^n x \cos^m x\,\mathrm{d}x$

  1. *If n is odd use substitution $u = \cos x$ (Strip out one sine and convert the rest to cosine.)*
  2. *If m is odd use substitution $u = \sin x$ (Strip out one cosine and convert the rest to sine.)*
  3. *If both n and m are odd use 1 or 2.*
  4. *If both n and m are even, use the half-angle identities:*

$$\sin^2 x = \frac{1}{2}(1-\cos 2x); \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x); \qquad \sin x \cos x = \frac{1}{2}\sin 2x$$

- How to evaluate $\int \sin(Ax)\cos(Bx)\,\mathrm{d}x$, $\int \sin(Ax)\sin(Bx)\,\mathrm{d}x$, $\int \cos(Ax)\cos(Bx)\,\mathrm{d}x$

  Use the following identities:(The identities below need not be memorized for the exam)

$$\sin A \cos B = \tfrac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$$

$$\sin A \sin B = \tfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$\cos A \cos B = \tfrac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$$

8.3: Trigonometric Substitutions

integral with	substitution	identity
$a^2 - x^2$	$x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$