Math 172 Exam 3
KEY POINTS (sections 9.2, 10.1-10.6)

9.2: First-Order Linear Differential Equations

- A first order ODE is called **linear** if it is expressible in the form

$$y' + p(t)y = g(t) \tag{1}$$

where $p(t)$ and $g(t)$ are given functions.

- **The Method of Integrating Factors**

Step 1 Put ODE in the form (1).

Step 2 Find the integrating factor

$$\mu(t) = e^{\int p(t)\mathrm{d}t}$$

Note: Any μ will suffice here, thus take the constant of integration $C = 0$.

Step 3 Multiply both sides of (1) by μ and use the Product Rule for the left side to express the result as

$$(\mu(t)y(t))' = \mu g(x) \tag{2}$$

Step 4 Integrate both sides of (2). Note: Be sure to include the constant of integration in this step!

Step 5 Solve for the solution $y(t)$.

10.1: Sequences

- If $\lim\limits_{n\to\infty} a_n$ exists and finite then we say that the sequence $\{a_n\}$ **converges**. Otherwise, we say the sequence **diverges**. (Recall all techniques for finding limits at infinity.)

- The *Squeeze* Theorem for Sequences: If $a_n \leq b_n \leq c_n$ for all n and the sequences $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \to \infty$, then $\lim\limits_{n\to\infty} b_n = L$.

- If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

- $\{a_n\}$ *increasing:* show that $a_{n+1} - a_n > 0$, or $f'(x) > 0$ (where $f(n) = a_n$); or $\dfrac{a_n + 1}{a_n} > 1$ (provided $a_n > 0$ for all n.) Note: reverse signs for $\{a_n\}$ *decreasing.*

10.2: Series

- Infinite series $\sum\limits_{n=1}^{\infty} a_n$ ($n = 1$ for convenience, it can be anything).

- Partial sums: $s_N = \sum\limits_{n=1}^{N} a_n$. Note $s_N = s_{N-1} + a_N$.

- If $\{s_N\}_{N=1}^{\infty}$ is convergent and $\lim_{N\to\infty} s_N = s$ exists as a real number, then the series $\sum_{n=1}^{n} a_n$ is *convergent*. The number s is called the **sum** of the series.
- Series we can sum:
  - Geometric Series $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad -1 < r < 1$
  - Telescoping Series
- THE TEST FOR DIVERGENCE: *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*
- **The Test for Divergence cannot be used to prove that a series converges. It can only show a series is divergent.**

10.3 : The Integral and Comparison Tests; Estimating Sums

THE TEST FOR DIVERGENCE: *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum a_n$ *is divergent.*	If $\lim_{n\to\infty} a_n = 0$ then the series may or may not converge.
THE INTEGRAL TEST *Let* $\sum a_n$ *be a* **positive** *series. If* f *is a continuous and decreasing function on* $[a, \infty)$ *such that* $a_n = f(n)$ *for all* $n \geq a$ *then* $\sum a_n$ *and* $\int_a^{\infty} f(x)\,dx$ *both converge or both diverge.*	Apply to positive series only when $f(x)$ is easy to integrate.
THE COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with* **nonnegative** *terms and* $a_n \leq b_n$ *for all* n. • *If* $\sum b_n$ *is convergent then* $\sum a_n$ *is also convergent.* • *If* $\sum a_n$ *is divergent then* $\sum b_n$ *is also divergent.*	• It applies to series with non-negative terms only. • Try it as a last resort (other tests are often easier to apply). • It requires some skills in chosing a series for comparison.
LIMIT COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with* **positive** *terms . If* $$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$ *where* c *is a finite number and* $c > 0$*, then either both series converge or both diverge.*	• It applies to positive series only. • It requires less skills to choose series for comparison than in Comparison test.

- *FACT:* **The p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges if $p > 1$ and diverges if $p \le 1$.**(by Integral Tests)
- REMAINDER ESTIMATE FOR THE INTEGRAL TEST
  If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \le R_n \le \int_{n}^{\infty} f(x)\,\mathrm{d}x$$

10.4 : Other Convergence Tests

ALTERNATING SERIES TEST: *If $b_n > 0$, $\lim_{n\to\infty} b_n = 0$ and the sequence $\{b_n\}$ is decreasing then the series $\sum(-1)^n b_n$ is convergent.*	It applies only to alternating series.
RATIO TEST For a series $\sum a_n$ with nonzero terms define $L = \lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert$. • If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.) • If $L > 1$ then the series is divergent. • If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).	• Try it when a_n involves factorials or n-th powers. • The series need not have positive terms and need not be alternating to use it. • Absolute convergence implies convergence.

The Alternating Series Theorem. *If $\sum_{n=1}^{\infty}(-1)^n b_n$ is a convergent alternating series and you used a partial sum s_n to approximate the sum s (i.e. $s \approx s_n$) then $|R_n| \le b_{n+1}$.*

10.5: Power Series

- For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only 3 possibilities:
  1. There is $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. We call such R the **radius of convergence.**
  2. The series converges only for $x = a$ (then $R = 0$).
  3. The series converges for all x (then $R = \infty$).
- We find the radius of convergence using the **Ratio Test.**
- An **interval of convergence** is the interval of all x's for which the power series converges.
- You must check the endpoints $x = a \pm R$ individually to determine whether or not they are in the interval of convergence.

10.6: Representation of Functions as Power Series

Key Points

- Geometric Series Formula:

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

- **Term-by-term Differentiation and Integration of power series:**

  If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

  - $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
  - $\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$

  The radii of convergence of the power series for $f'(x)$ and $\int f(x)\,\mathrm{d}x$ are both R.