

Math 172

KEY POINTS (sections 10.7&10.9)

Note: Review Key Points for Exams I,II,III.

10.7: Taylor and Maclaurin Series

- The Taylor series for $f(x)$ about $x = a$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \end{aligned}$$

- The Maclaurin series is the Taylor series about $x = 0$ (i.e. $a=0$):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

- Known Maclaurin series and their intervals of convergence you must have memorized:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad (1, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (-\infty, \infty)$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad [-1, 1]$$

10.9: Applications of Taylor Polynomials

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$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \underbrace{\sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n}_{T_N(x)} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n}_{R_N(x)}$$

N - th degree

Remainder

Taylor polynomial

- How to estimate the Remainder $|R_N(x)| = |f(x) - T_N(x)|$?
 - Using graph.
 - If the series happens to be an alternating series, you can use the Alternating Series Theorem.
 - In all cases you can use **Taylor's Inequality**:

$$|R_N(x)| \leq \frac{M}{(N+1)!} |x-a|^{N+1}$$

where $|f^{(N+1)}(x)| \leq M$ for all x in an interval containing a .