2. Sets

2.1&2.2: Sets and Subsets. Combining Sets.

- **Set Terminology and Notation**

  DEFINITIONS:

  Set is well-defined collection of objects.

  Elements are objects or members of the set.

  - **Roster notation:**

    $A = \{a, b, c, d, e\}$ Read: Set A with elements a, b, c, d, e.

  - **Indicating a pattern:**

    $B = \{a, b, c, ..., z\}$ Read: Set B with elements being the letters of the alphabet.

  Set-builder notation:

  $B = \{x|x \text{ is a letter of the English alphabet}\}$

  $C = \{x|x \text{ is a student in this classroom}\}$

  The symbol "|" is read "such that".

  If a is an element of a set A, we write $a \in A$ that read "a belongs to A." However, if a does not belong to A, we write $a \notin A$.

Very common sets:

- $\mathbf{R}$ is the set of all *real* numbers;
- $\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of all *integers*;
- $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of all *positive integers*;
- $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of all *natural* numbers;
- $\mathbf{Q}$ is the set of all *rational numbers*;

Other sets:

- $\mathbf{E}$ is the set of all *even integers*;
- $\mathbf{O}$ is the set of all *odd integers*;
- $n\mathbf{Z}$ is the set of all integers multiples of n $(n \in \mathbf{Z})$;

Application to set definitions

Set-builder notation:

$A = \{x \in S | P(x)\}$ is the set of all elements x in S such that the open sentence $P(x)$ is a true statement.

EXAMPLE 1. *Use the set-builder notation to describe the following sets:*

a) $\mathbf{N}$

b) $\mathbf{Q}$

c) $\mathbf{O}$

d) $\mathbf{E}$

e) $5\mathbf{Z}$

Interval notation:

EXAMPLE 2. **a)** $\{n \in \mathbf{Z} |\ 3 \le n < 10\} =$

b) $\{x \in \mathbf{R} |\ (x \ge 0) \wedge (x \in \mathbf{Z})\} =$

c) $\{x \in \mathbf{R} |\ -2015 \le x \le 2016\} =$

d) $\{x \in \mathbf{R} |\ |x+5| \le 7\} =$

Intervals:

- bounded intervals:

1. closed interval $[a, b] =$
2. open interval $(a, b) =$
3. half-open,half-closed interval $(a, b] =$
4. half-closed,half-open interval $[a, b) =$

- unbounded intervals:

5. $[a, \infty) =$
6. $(a, \infty) =$
7. $(-\infty, a] =$
8. $(-\infty, a) =$
9. $(-\infty, \infty) =$

infinite set

finite set

cardinality of A, $|A|$

Subsets

Two sets, A and B, are **equal**, written $A = B$, if and only if they have exactly the same elements. (NOTE: they do not have to be in the same order!). If every element in set A is also an element in set B, then A is a subset of B, written $A \subseteq B$. If $A \subseteq B$, but $A \neq B$, then A is a **proper** subset of B, written $A \subset B$.

EXAMPLE 3. *Given*

$$\begin{array}{l} A = \{a, e, i, o, u\} \\ B = \{u, i, e, a, o\} \\ C = \{a, e, i, o\} \\ D = \{e, i, o, a\} \end{array}$$

. *Then*

EXAMPLE 4. *Which of the following are TRUE?*

1. $\mathbf{Z}^+ \subset \mathbf{Z}$
2. $\mathbf{Z}^+ \subseteq \mathbf{Z}$
3. $\mathbf{N} \subseteq \mathbf{Z}^+$
4. $\mathbf{Z} \subset \mathbf{Q} \subseteq \mathbf{R}$

The **empty set** is the set that doesn't have any elements, denoted by $\emptyset$ or $\{\}$. The **universal set** is the set that contains all of the elements for a problem, denoted by U.

EXAMPLE 5. *Give all the subsets for these sets.*

(a) $A = \{0, 1\}$ **(b)** $X = \{a, b, y\}$

EXAMPLE 6. *List all elements of the following set:*

$$\{x \in \mathbf{R} \mid \sin x = 2\}$$

EXAMPLE 7. *Given* $A = \{0, 1, 2, \ldots, 8\}$, $B = \{1, 3, 5, 7\}$, $C = \{3, 5, 1, 7\}$, *and* $D = \{5, 3, 1\}$, $E = \emptyset$ *then which of the following are TRUE?*

(**a**)$B = C$ (**b**)$B \subseteq C$ (**c**)$B \subset C$ (**d**)$C \subseteq B$ (**e**)$D \subset B$

(**f**)$D \subseteq B$ (**g**)$B \subset D$ (**h**)$8 \in A$ (**i**) $\{4, 6\} \subset A$ (**j**)$1, 5 \subset A$

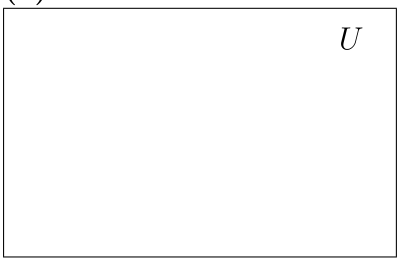
(**k**)$9 \notin C$ (**l**)$D \subseteq D$ (**m**)$\emptyset = 0$ (**n**)$0 \in E$ (**o**)$A \in A$

VENN DIAGRAMS - a visual representation of sets (the universal set U is represented by a

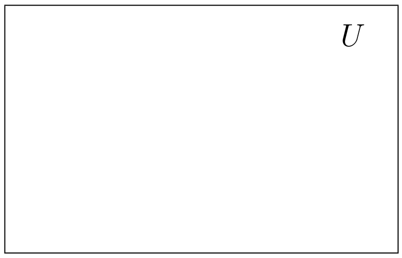
rectangle, and subsets of U are represented by regions lying inside the rectangle).

EXAMPLE 8. *Use Venn diagrams to illustrate the following statements:*

(**a**) $A = B$



(**b**) $A \subset B \subset C$

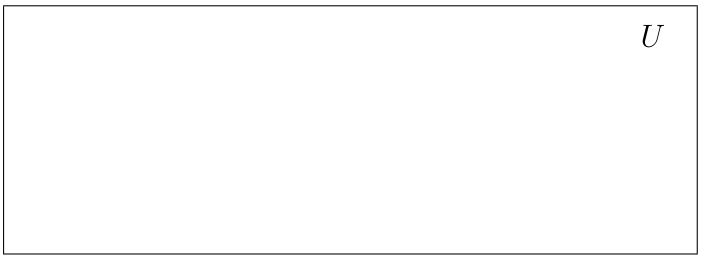


(**c**) *A and B are not subsets of each other.*

• OPERATIONS OF SETS

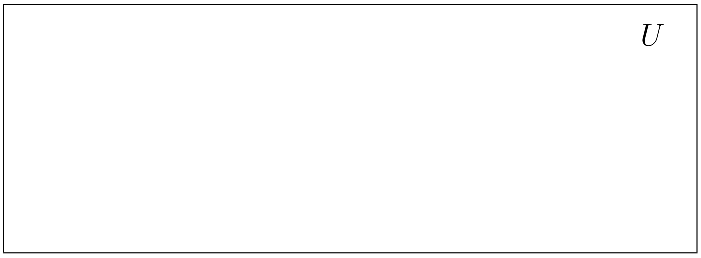
Let A and B be sets. The **union** of A and B, written $A \cup B$, is the set of all elements that belong to either A or B or both. Symbolically:

$$A \cup B = \{x | x \in A \vee x \in B\}.$$



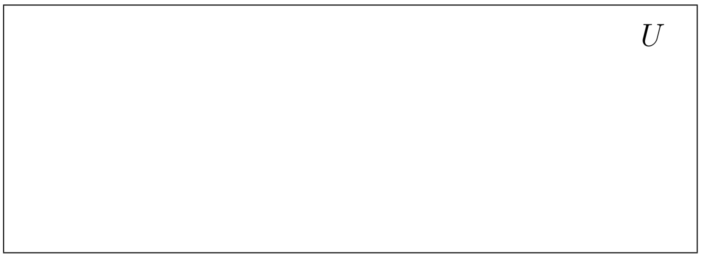
Let A and B be sets. The **intersection** of A and B, written $A \cap B$, is the set of all elements in common with A and B. Symbolically:

$$A \cap B = \{x | x \in A \ \wedge \ x \in B\}.$$



DEFINITION 9. *Let A and B be sets. The* **complement of** A **in** B *denoted $B - A$, is* $\{b \in B | b \notin A\}$.

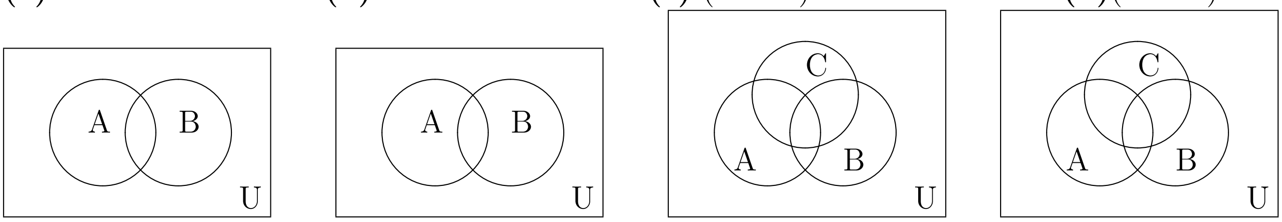
REMARK 10. For convenience, if U is a universal set and A is a subset in U, we will write $U - A = \bar{A}$, called simply the **complement** of A.



EXAMPLE 11. *Find $\bar{U}$ and $\bar{\emptyset}$.*

EXAMPLE 12. *Shade the Venn diagrams below to represent the following sets*

(a) $A \cup \bar{B}$ **(b)** $\bar{A} \cap B$ **(c)** $(A \cap B) \cup C$ **(d)**$\overline{(A \cup B)} \cap C$



EXAMPLE 13. *Use these sets to find the following:* $U = \{0, 1, 2, ..., 9, 10\}$, $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{1, 3, 5, 7, 9\}$, $C = \{0, 3, 4, 5, 7\}$, $D = \{0, 6, 7, 9\}$, $E = \{1, 3, 8, 9\}$

(a) $B \cup D$

(b) $A \cup B$

(c) $\bar{C}$

(d) $D - E$

(e) $E - D$

EXAMPLE 14. *Use set notation to reformulate the following theorem: "Every real-valued continuous function on* $[a, b]$ *is integrable on* $[a, b]$*." Also describe a universal set. Discuss the converse statement.*

THEOREM 15. *Let A and B be sets contained in some universal set U. Then $A \subseteq B$ if and only if $\bar{B} \subseteq \bar{A}$.*

Proving set properties

Use the following tautologies:

- $x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$
- $x \in A \cup B \Leftrightarrow$
- $x \in A - B \Leftrightarrow$
- $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$
- $A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B)$

Question: Let $A = \{n \in \mathbf{Z} | n \text{ is even}\}$ and $B = \{n \in \mathbf{Z} | n^2 \text{ is even}\}$. Are these sets the same?

Methods:

- To prove $A \subseteq B$ it is sufficient to prove $x \in A \Rightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $x \in A \Leftrightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $A \subseteq B$ and $B \subseteq A$.
- To show that $A = \emptyset$ it is sufficient to show that $x \in A$ implies a false statement.

EXAMPLE 16. *Let* $A = \{n \in \mathbb{Z} | n = 3t - 2$ *for some* $t \in \mathbb{Z}\}$ *and* $B = \{n \in \mathbb{Z} | n = 3t + 1$ *for some* $t \in \mathbb{Z}\}$. *Prove that* $A = B$.

EXAMPLE 17. *Let* A *and* B *be sets. Show that* $(A - B) \cap B = \emptyset$.

PROPOSITION 18. *Let* $A, B,$ *and* C *be sets, and suppose* $A \subseteq B$ *and* $B \subseteq C$. *Then* $A \subseteq C$.

Fundamental properties of sets

THEOREM 19. *The following statements are true for all sets A, B, and C.*

1. $A \cup B = B \cup A$ (`commutative`)
2. $A \cap B = B \cap A$ (`commutative`)
3. $(A \cup B) \cup C = A \cup (B \cup C)$ (`associative`)
4. $(A \cap B) \cap C = A \cap (B \cap C)$ (`associative`)
5. $A \subseteq A \cup B$.
6. $A \cap B \subseteq A$.
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (`distributive`)
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (`distributive`)
9. *The empty set is a subset of every set (i.e. $\emptyset \subset A$).*
10. $A \cup \emptyset = A$.
11. $A \cap \emptyset = \emptyset$.

    `DeMorgan's Laws:` *If A and B are the sets contained in some universal set U then*

12. $\overline{A \cup B} = \bar{A} \cap \bar{B}$.
13. $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

EXAMPLE 20. *Let A and B be subsets of a universal set U. Then $A \subseteq B \Leftrightarrow A \cup B = B$.*

(a) *Criticize the proposed "proof" of the above result:*

> *Let $x \in A \subseteq B$. Then $x \in A$ and B, so $x \in A \cup B$ and $A \cup B = B$.*
> *Let $x \in A \cup B = B$ then $x \in A$ or $x \in B$ and $x \in B$. Therefore $A \subseteq B$.*

(b) *Prove the above result.*

Cartesian Product

DEFINITION 21. *Let A and B be sets. The* **Cartesian product** *of A and B, written $A \times B$, is the following set:*

$$A \times B = \{(a, b) | \ a \in A \wedge b \in B\}.$$

Informally, $A \times B$ is the set of **ordered** pairs of objects.

EXAMPLE 22. *Given $A = \{0, 1\}$ and $B = \{4, 5, 6\}$.*

(a) *Does the pair $(6, 1)$ belong to $A \times B$?*

(b) *List the elements of $A \times B$.*

(c) *What is the cardinality of $A \times B$?*

(d) *List the elements of $A \times A \times A$.*

(e) *Does the triple $(1, 6, 4)$ belong to $A \times B \times B$?*

(f) *Describe the following sets $R \times R$, $R \times R \times R$.*

2.3 Collections of Sets

• **Power set**

DEFINITION 23. *Let A be a set. The power set of A, written $P(A)$, is*

$$P(A) = \{X \mid X \subseteq A\}.$$

EXAMPLE 24. *Let $A = \{-1, 0, 1\}$.*

1. *Write all subsets of A.*

2. *Find all elements of power set of A.*

3. *Find $|P(A)|$.*

4. *Write 3 subsets of $P(A)$.*

5. *Find $|P(P(A))|$.*

6. *Find $P(\emptyset)$ and $P(\{-1\})$.*

• **Indexed Sets**

DEFINITION 25. *Let I be a set. An indexed collection of set $\{A_\alpha\}_{\alpha\in I}$ represents a collection of sets such that for every $\alpha \in I$, there is a corresponding set A_α. In this case we call I the* **indexed set.**

• **Union and Intersection**

EXAMPLE 26. *Negate the following statement forms.*

(a) $x \in \bigcup_{\alpha\in I} A_\alpha \Leftrightarrow \exists \alpha \in I \ni x \in A_\alpha$

(b) $x \in \bigcap_{\alpha\in I} A_\alpha \Leftrightarrow \forall \alpha \in I, x \in A_\alpha$

EXAMPLE 27. *Given $B_i = \{i, i+1\}$ for $i = 1, 2, \ldots, 10$. Determine the following*

(a) $\bigcap_{i=1}^{10} B_i$

(b) $B_i \cap B_{i+1}$

(c) $\bigcap_{i=k}^{k+1} B_i$ *where* $1 \le k < 10$.

(d) $\bigcap_{i=j}^{k} B_i$ *where* $1 \le j < k \le 10$.

EXAMPLE 28. $A_n = \left\{x \in \mathbf{R} \mid -\frac{1}{n} \le x \le \frac{1}{n}, \quad n \in \mathbf{Z}^+\right\}$. *Find* $\bigcup_{n\in\mathbf{Z}^+} A_n$ *and* $\bigcap_{n\in\mathbf{Z}^+} A_n$.

- **Partitions of sets**

Disjoint sets

Mutually exclusive

DEFINITION 29. *A partition* $\mathcal{P}$ *of* A *is a subset of the power set* $P(A)$ *such that*

1. $X \in \mathcal{P} \Rightarrow X \neq \emptyset$
2. $\bigcup_{X\in\mathcal{P}} X = A$
3. *If* $X, Y \in \mathcal{P}$ *and* $X \neq Y$ *then* $X \bigcap Y = \emptyset$.

EXAMPLE 30. *Determine which of these sets are partitions of* $A = \{1, 2, 3, 4, 5, 6\}$*:*

$$\mathcal{P}_1 = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\}$$

$$\mathcal{P}_2 = \{\{1, 2, 3\}, \{4\}, \emptyset, \{5, 6\}\}$$

$$\mathcal{P}_3 = \{\{1, 2\}, \{3, 4, 5\}, \{5, 6\}\}$$

$$\mathcal{P}_4 = \{\{1, 4\}, \{3, 5\}, \{2\}\}$$

EXAMPLE 31. *Let* $A = 2\mathbf{Z}$ *and* $B = \{n \in \mathbf{Z} | n = 2t + 1 \text{ for some } t \in \mathbf{Z}\}$. *Show that* $\{A, B\}$ *is a partition of* $\mathbf{Z}$.

EXAMPLE 32. *Give an example of a partition for* $A = \{1, 2, \ldots, 12\}$ *such that* $|\mathcal{P}| = 5$.

EXAMPLE 33. *Give an example of a partition for* $\mathbf{R}$.

THEOREM 34. *Let* A *and* B *be finite disjoint sets. Then*

$$|A \cup B| = |A| + |B|.$$

COROLLARY 35. *Let* A *and* B *be finite sets. Then*

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

COROLLARY 36. *Let* $A_1, A_2, \ldots, A_n$ *be a collection of finite mutually disjoint sets. Then*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|.$$