

4. Sets

4.1. The language of sets

- **Set Terminology and Notation**

Set is a well-defined collection of objects.

Elements are objects or members of the set.

Describing a Set

- **Roster notation:**

$A = \{a, b, c, d, e\}$ Read: Set A with elements a, b, c, d, e .

- **Indicating a pattern:**

$B = \{a, b, c, \dots, z\}$ Read: Set B with elements being the letters of the alphabet.

If a is an element of a set A , we write $a \in A$ that read "a belongs to A ." However, if a does not belong to A , we write $a \notin A$.

Set-builder notation (a more precise way of describing a set)

NOTATION 1. Let $P(x)$ be a predicate. Then the notation

$$A = \{x|P(x)\} \quad \text{or} \quad A = \{x : P(x)\}$$

denotes the set A of all elements x such that $P(x)$ is a true statement. In symbols,

$$\forall x, x \in A \Leftrightarrow P(x).$$

When D is a set containing the set A , the notation

$$A = \{x \in D | P(x)\} = \{x | x \in D \wedge P(x)\}$$

denotes the set A of all elements in D such that $P(x)$ is a true statement. In symbols,

$$\forall x \in D, x \in A \Leftrightarrow P(x).$$

EXAMPLE 2. Use set-builder notation to describe the following sets in two different ways:

a) \mathbf{O}

b) $5\mathbf{Z}$

c) \mathbf{N}

d) \mathbf{Q}

EXAMPLE 3. Rewrite the following sets using roster notation:

$$A = \{x|x \in \mathbf{R} \wedge |x| = 1\} =$$

$$B = \{x|x \in \mathbf{R} \wedge x^4 = 1\} =$$

$$C = \{x|x \in \mathbf{C} \wedge x^4 = 1\} =$$

Interval notation:**Intervals**

NOTATION 4. • *bounded intervals:*

1. *closed interval* $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

2. *open interval* $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

3. *half-open, half-closed interval* $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

4. *half-closed, half-open interval* $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

• *unbounded intervals:*

5. $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$

6. $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$

7. $(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$

8. $(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$

9. $(-\infty, \infty) = \mathbb{R}$

EXAMPLE 5. *Represent the following sets in interval notation when it is possible.*

a) $\{x \in \mathbf{R} \mid (x \geq 0) \wedge (x \in \mathbf{Z})\} =$

b) $\{x \in \mathbf{Z} \mid 3 \leq x < 10\} =$

c) $\{x \in \mathbf{R} \mid -2018 \leq x \leq 2019\} =$

Subsets

- Two sets, A and B, are **equal**, written $A = B$, if and only if they have exactly the same elements. (NOTE: they do not have to be in the same order!).

$$\{a, b, c\} \quad \{c, a, b\} \quad \{a, b, c, b\}$$

- If every element in set A is also an element in set B, then A is a subset of B, written $A \subseteq B$.

Note that $A \subseteq A$.

- If $A \subseteq B$, but $A \neq B$, then A is a **proper** subset of B, written $A \subset B$.

$$A \subseteq B \Leftrightarrow (A \subset B \quad \wedge \quad A = B)$$

and

$$A \subset B \Leftrightarrow (A \subseteq B \quad \wedge \quad A \neq B)$$

- The **empty set** is the set that doesn't have any elements, denoted by \emptyset or $\{\}$.
- The **universal set** is the set that contains all of the elements for a problem, denoted by U .

Using Symbols

- $A \subseteq B \Leftrightarrow \forall x, (x \in A \Rightarrow x \in B)$
- $A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A)$
- $A = B \Leftrightarrow \forall x, (x \in A \Leftrightarrow x \in B)$
- $A \neq B \Leftrightarrow$

EXAMPLE 6. *Prove or disprove: “If $B = \{x|x \in \mathbf{R} \wedge x^4 = 1\}$ and $C = \{x|x \in \mathbf{C} \wedge x^4 = 1\}$, then $B = C$.”*

EXAMPLE 7. *Let $A = \{n \in \mathbf{Z}|n \text{ is even}\}$, $B = \{n \in \mathbf{Z}|n^2 \text{ is even}\}$, and $C = \{n^2|n \text{ is even}\}$. Are these sets the same?*

EXAMPLE 8. *Let $A = \{n \in \mathbf{Z}|n = 3t - 2 \text{ for some } t \in \mathbf{Z}\}$ and $B = \{n \in \mathbf{Z}|n = 3t + 1 \text{ for some } t \in \mathbf{Z}\}$. Prove that $A = B$.*

Cardinality

infinite sets $\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [1, 3), \{2^n | n \in \mathbb{N}\}$

finite sets $\{\Delta, \square\}, \{2^n | n \in \{3, 4, 5\}\}$

cardinality of A , $|A|$

$$|\emptyset| = \quad , \quad |\{x \in \mathbb{R} | x^4 = 1\}| =$$

EXAMPLE 9. Let A and B be two sets.

(a) **TRUE/FALSE** If $A = B$, then $|A| = |B|$.

(b) **TRUE/FALSE** If $|A| = |B|$, then $A = B$.

4.2 Operations on sets

VENN DIAGRAMS

- a visual representation of sets (the universal set U is represented by a rectangle, and subsets of U are represented by regions lying inside the rectangle).

EXAMPLE 10. Use Venn diagrams to illustrate the following statements:

(a) $A = B$



(b) $A \subset B \subset C$



(c) A and B are not subsets of each other.



DEFINITION 11. Let A and B be sets in a universal set U . The **union** of A and B , written $A \cup B$, is the set of all elements that belong to either A or B or both. Symbolically:

$$A \cup B = \{x \in U | x \in A \vee x \in B\}.$$

DEFINITION 12. Let A and B be sets in a universal set U . The **intersection** of A and B , written $A \cap B$, is the set of all elements in common with A and B . Symbolically:

$$A \cap B = \{x \in U | x \in A \wedge x \in B\}.$$

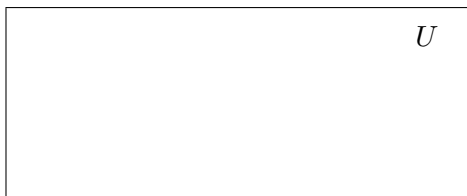


DEFINITION 13. Let A and B be sets. The **complement of A in B** denoted $B - A$, is

$$B - A = \{x \in U | x \in B \wedge x \notin A\}$$



REMARK 14. For convenience, if U is a universal set and A is a subset in U , we will write $U - A = \bar{A}$, called simply the **complement** of A .



EXAMPLE 15. Let $U = \{0, 1, 2, \dots, 9, 10\}$ be a universal set, $A = \{0, 2, 4, 6, 8, 10\}$, and $B = \{1, 3, 5, 7, 9\}$. Find

$$\overline{(A \cap B)} \cap \overline{(A \cup B)}.$$

set notation	=	\subset, \subseteq	\cup	\cap	$\bar{\square}$	\emptyset
logical connectivity						

Power set

DEFINITION 16. Let A be a set. The power set of A , written $P(A)$, is the following set

$$P(A) = \{X | X \subseteq A\}.$$

EXAMPLE 17. Find the following

(a) $P(\{x, y\})$

(b) $|P(\{x, y\})|$

EXAMPLE 18. Let $A = \{-1, 0, 1\}$.

1. Find all elements of power set of A .
2. What is the number of subsets of A ? What is the number of subsets of $P(A)$?
3. Find $|P(A)|$ and $|P(P(A))|$
4. Write 3 subsets of A and 5 subsets of $P(A)$.
5. What are $|P(A)|$ and $|P(P(A))|$ for an arbitrary set A ?

EXAMPLE 19. Find

- (a) $P(\{\Delta\})$
- (b) $P(\emptyset)$
- (c) $P(P(\emptyset))$
- (d) $P(\{\Delta, \square\})$
- (e) $P(\{\emptyset, \{\emptyset\}\})$

REMARK 20. Note that

$$\emptyset \subseteq \{\emptyset, \{\emptyset\}\}, \quad \emptyset \subset \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset\} \in \{\emptyset, \{\emptyset\}\},$$

as well as

$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}, \quad \{\{\emptyset\}\} \notin \{\emptyset, \{\emptyset\}\}, \quad \{\{\emptyset\}\} \in P(\{\emptyset, \{\emptyset\}\}).$$

Proofs Involving Sets

Proving set properties

Use the following tautologies:

- $x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$
- $x \in A \cup B \Leftrightarrow (x \in A \vee x \in B)$
- $x \in \bar{A} \Leftrightarrow x \notin A$
- $x \in A - B \Leftrightarrow (x \in A \wedge x \notin B)$
- $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$
- $A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B)$
- $(x, y) \in A \times B \Leftrightarrow (x \in A \wedge y \in B)$

Methods:

- To prove $A \subseteq B$ it is sufficient to prove $x \in A \Rightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $x \in A \Leftrightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $A \subseteq B$ and $B \subseteq A$.
- To show that $A = \emptyset$ it is sufficient to show that $x \in A$ implies a false statement.

Fundamental properties of sets

THEOREM 23. *The following statements are true for all sets A , B , and C .*

1. $A \cup B = B \cup A$ (commutative)
2. $A \cap B = B \cap A$ (commutative)
3. $(A \cup B) \cup C = A \cup (B \cup C)$ (associative)
4. $(A \cap B) \cap C = A \cap (B \cap C)$ (associative)
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (distributive)
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (distributive)

DeMorgan's Laws: *If A and B are the sets contained in some universal set U then*

7. $\overline{A \cup B} = \bar{A} \cap \bar{B}$.
8. $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

THEOREM 24. *Let A and B be a subsets of a universal set U . Then*

1. $\overline{\overline{A}} = A$.

2. $\overline{\emptyset} = U$.

3. $\overline{U} = \emptyset$

4. $A \subseteq A \cup B$.

5. $A \cap B \subseteq A$.

6. *The empty set is a subset of every set. (Namely, for every set A , $\emptyset \subseteq A$. If $A \neq \emptyset$, then $\emptyset \subset A$.).*

7. $A \cup \emptyset = A$.

8. $A \cap \emptyset = \emptyset$.

9. $A \cap A = A \cup A = A$

EXAMPLE 25. Let A and B be subsets of a universal set U . Show that $(A - B) \cap B = \emptyset$.

PROPOSITION 26. Let A and B be subsets of a universal set U . Then

$$A - B = A \cap \bar{B}.$$

EXAMPLE 27. Let A, B and C be sets. Prove that

$$A - (B \cup C) = (A - B) \cap (A - C)$$

EXAMPLE 28. For the sets A, B and C prove that

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

PROPOSITION 29. *Let A, B , and C be sets, and suppose $A \subseteq B$ and $B \subseteq C$. Then $A \subseteq C$.*

EXAMPLE 30. *Let A, B, C and D be sets. If $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.*

EXAMPLE 31. *Prove the following statement. Let A and B be subsets of a universal set U . Then $A \subseteq B \Leftrightarrow A \cup B = B$.*

EXAMPLE 32. Let A and B be subsets of a universal set U . Prove that

$$A = A - B \Leftrightarrow A \cap B = \emptyset.$$

4.3 Arbitrary unions and intersections

DEFINITION 33. Let I be a set. An **indexed collection of sets** $\{A_\alpha\}_{\alpha \in I}$ represents a collection of sets such that for every $\alpha \in I$, there is a corresponding set A_α . In this case we call I the **indexed set**.

Collection of sets	Indexed set	Shortened notation
$A_0, A_1, A_2, A_3, \dots, A_{2016}$		
B_3, B_6, B_9, B_{77}		
$C_5, C_{10}, C_{15}, \dots, C_{2015}$		

• Union and Intersection

EXAMPLE 34. Complete the following

$$(a) \quad x \in \bigcup_{\alpha \in I} A_\alpha \Leftrightarrow \exists \alpha \in I \ni x \in A_\alpha$$

$$x \notin \bigcup_{\alpha \in I} A_\alpha \Leftrightarrow$$

$$(b) \quad x \in \bigcap_{\alpha \in I} A_{\alpha} \Leftrightarrow \forall \alpha \in I, x \in A_{\alpha}$$
$$x \notin \bigcap_{\alpha \in I} A_{\alpha} \Leftrightarrow$$

EXAMPLE 35. Given $B_i = \{i, i + 1\}$ for $i = 1, 2, \dots, 10$. Determine the following

$$(a) \quad \bigcap_{i=1}^{10} B_i$$

$$(b) \quad B_i \cap B_{i+1}$$

$$(c) \quad \bigcap_{i=k}^{k+1} B_i \text{ where } 1 \leq k < 10.$$

$$(d) \quad \bigcup_{i=j}^k B_i \text{ where } 1 \leq j < k \leq 10.$$

EXAMPLE 36. $A_n = \{x \in \mathbf{R} \mid -\frac{1}{n} \leq x \leq \frac{1}{n}\}$, $n \in \mathbf{Z}^+$. Find $\bigcup_{n \in \mathbf{Z}^+} A_n$ and $\bigcap_{n \in \mathbf{Z}^+} A_n$.