1.1 Language and Logic

Mathematical Statements

DEFINITION 1. *A* ***proposition*** *is any declarative sentence (i.e. it has both a subject and a verb) that is either true or false, but not both.*

A proposition cannot be neither true nor false and it cannot be both true and false.
A proposition is an example of mathematical statement. Sometimes it is called a statement.

EXAMPLE 2. *Determine whether the following sentences are propositions.*

1. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *The integer 5 is odd.*
2. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *The integer 24277151704311 is prime.*
3. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ $15 + 7 = 22$
4. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *Substitute the number* 7 *for* x.
5. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *What is the derivative of* $\cos x$*?*
6. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *Apple manufactures computers.*
7. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *I am telling a lie.*

- **Set Terminology and Notation (very short introduction[1])**
  Set is a well-defined collection of objects.
  Elements are objects or members of the set.

- **Roster notation:**

  $A = \{a, b, c, d, e\}$ Read: Set A with elements a, b, c, d, e.

- **Indicating a pattern:**

  $B = \{a, b, c, ..., z\}$ Read: Set B with elements being the letters of the alphabet.

If a is an element of a set A, we write $a \in A$ that read "a belongs to A." However, if a does not belong to A, we write $a \notin A$.

Some Number sets:

- $\mathbb{R}$ is the set of all *real* numbers;
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of all *integers*;
- $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of all *natural* numbers;
- $\mathbb{Q}$ is the set of all *rational numbers*;
- $\mathbb{E}$ is the set of all *even integers*;
- $\mathbb{O}$ is the set of all *odd integers*;
- $n\mathbb{Z}$ is the set of all integers multiples of n ($n \in \mathbb{Z}$);

[1] We will study SETS in Chapter 4!

Trichotomy Axiom *Given fixed real numbers a and b, exactly one of the statements $a < b$, $a = b$, $b < a$ is true.*

EXAMPLE 3. *Determine whether the following sentences are propositions.*

1. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ $|x| > 7$
2. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *The absolute value of the real number x is greater than 7.*
3. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *The absolute value of a real number x is greater than 7. (Hint: Rewriting in a different mode is useful!)*

A ***predicate*** *is any declarative sentence containing one or more variables, each variable representing a value in some prescribing set, called the* ***universe****, and which becomes a proposition when values from their respective universes are substituted for these variables.*

A predicate is another example of mathematical statement. Sometimes it is also called an open sentence.

EXAMPLE 4. *Complete:*

(a) *Let $P(x) : x + 5 = 7$ where $x \in \mathbb{R}$. Then*

- $P(2)$ *is* ______________________________
- $P(3)$, $P(-1)$, $P(5.6)$ *are* ______________________________.
- $P(x)$ *becomes a true proposition when we substitute for x the values from the set* ____________________ *For all other values of x, $P(x)$ is a* _______________ *proposition.*
- *Conclusion:* $P(x) : x + 5 = 7$ *where $x \in \mathbb{R}$ is* _______________

(b) *Let $P(n) : (n - 3)^2 \leq 1$ where $n \in \mathbb{Z}$. Then*

- $P(n)$ *becomes a true proposition when we substitute for n the values from the set* ____________________ *For all other values of n, $P(n)$ is a* _______________ *proposition.*
- *Conclusion:* $P(n) : (n - 3)^2 \leq 1$ *where $n \in \mathbb{Z}$ is* _______________

NEGATION

DEFINITION 5. *If P is a mathematical statement, then the* **negation/denial** *of P, written $\neg P$ (read "not P"), is the mathematical statement "P is false".*

Although $\neg P$ could always be expressed as

It is not the case that P.

there are usually better (useful) ways to express the statement $\neg P$.

EXAMPLE 6. *Negate the following propositions in a useful way.*

1. P : *The integer 7 is even.*

____

____

____

2. $P:\quad 5^3 = 120 \qquad \neg P:$______________

3. P : *The absolute value of the real number x is less than* 5.

____

Basic Connectivities

We have two types of mathematical statements: propositions and predicates. We can build more complicated (compound) statements using the following logical connectivities:

$$\wedge, \quad \vee, \quad \neg, \quad \Rightarrow .$$

Logical connectivity	write	read	meaning
Conjunction	P$\wedge Q$	P and Q	Both P and Q are true
Disjunction	P$\vee Q$	P or Q	P is true or Q is true

P: Ben is a student.
Q: Ben is a grader.

TRUTH TABLES

P	Q	$P\wedge Q$	$Q\wedge P$	$P\vee Q$	$Q\vee P$

EXAMPLE 7. *Rewrite the following predicates (over $\mathbb{R}$) using disjunction or conjunction.*

(a) $P(x):\quad |x| \geq 10.$

(b) $P(x):\quad |x| < 10.$

Implications

DEFINITION 8. *Let P and Q be statements. The* **implication** $P \Rightarrow Q$ *(read "P implies Q") is the statement "If P is true, then Q is true."*

In implication $P \Rightarrow Q$, P is called *assumption, or hypothesis, or antecedent;* and Q is called *conclusion or consequent.*

EXAMPLE 9. P : *You pass the final exam.*
Q: *You pass the course.*

$P \Rightarrow Q$:

The truth table for implication:

P	Q	$P \Rightarrow Q$
T	T	
T	F	
F	T	
F	F	

Alternative Expressions for $P \Rightarrow Q$. Necessary and Sufficient Conditions

If P, then Q.	P implies Q.	P only if Q.	P is sufficient for Q.
	Q if P.	Q when P.	Q is necessary for P.
	Q whenever P.	Q, provided that P.	Whenever P, then also Q.

REMARK 10. Note however, if $P \Rightarrow Q$ is true, then it is not necessary that P is true in order for Q to be true. Even if Q is true, P may be false.

Converse and Contrapositive

DEFINITION 11. *The statement $Q \Rightarrow P$ is called a* **converse** *of the statement $P \Rightarrow Q$.*

DEFINITION 12. *The statement $(\neg Q) \Rightarrow (\neg P)$ is called a* **contrapositive** *of the statement $P \Rightarrow Q$.*

Biconditional "$\Leftrightarrow$"

For statements P and Q,

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

is called the **biconditional** of P and Q and is denoted by $P \Leftrightarrow Q$. The biconditional $P \Leftrightarrow Q$ is stated as

"*P is equivalent to Q.*" or "*P if and only if Q.*" (or "*P iff Q.*")
or as "*P is a necessary and sufficient condition for Q.*"

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T			
T	F			
F	T			
F	F			

EXAMPLE 13. *Complete:*

(a) *The biconditional* "The number 17 is odd if and only if 57 is prime." *is* ____________.

(a) *The biconditional* "The number 24 is even if and only if 17 is prime." *is* ____________.

(a) *The biconditional* "The number 17 is even if and only if 24 is prime." *is* ____________.

Logical Equivalence

DEFINITION 14. *Two compound statements are* **logically equivalent** *(write "$\equiv$") if they have the same truth tables, which means they both are true or both are false.*

Question: Are the statements $P \Rightarrow Q$ and $Q \Rightarrow P$ logically equivalent? __________

Some Fundamental Properties of Logical Equivalence

THEOREM 15. *For the statement forms P, Q and R,*

1. $\neg(\neg P) \equiv$

2. *Commutative Laws*
   $P \vee Q \equiv$
   $P \wedge Q \equiv$

3. *Associative Laws*
   $P \vee (Q \vee R) \equiv$
   $P \wedge (Q \wedge R) \equiv$

4. *Distributive Laws*
   $P \vee (Q \wedge R) \equiv$
   $P \wedge (Q \vee R) \equiv$

5. *De Morgan's Laws*
   $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$
   $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$

6. $\neg(P \Rightarrow Q) \equiv P \wedge (\neg Q)$.

7. $P \Rightarrow Q \equiv (\neg P) \vee Q$

8. $P \Rightarrow Q \equiv (\neg Q) \Rightarrow (\neg P)$

9. $P \Rightarrow Q$ *is NOT logically equivalent to* $Q \Rightarrow P$

Proof. Each part of the theorem is verified by means of a truth table or/and by a deductive proof.

P	Q					

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$\neg Q$	$P \wedge (\neg Q)$	
T	T					
T	F					
F	T					
F	F					

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$

Tautologies and Contradictions

Tautology: statement that is always true
Contradiction: statement that is always false

P	$\neg P$	$P \vee (\neg P)$	$P \wedge (\neg P)$
T			
F			

Methods to verify tautology/contradiction: truth table and deductive proof.

EXAMPLE 16. *Determine whether the following formula for the statements P and Q is a tautology, contradiction, or neither.*

$$\neg(P \Rightarrow Q) \Leftrightarrow P \wedge (\neg Q).$$

REMARK 17. Let P and Q be statements. The biconditional $P \Leftrightarrow Q$ is a tautology if and only if P and Q are logically equivalent.

Quantified Statements

EXAMPLE 18. *Consider the following* predicate *over* $\mathbb{N}$:

$$P(n): \ 4n+3 \text{ is prime.}$$

How to convert this predicate into a proposition with a truth value?

A predicate can be made into a proposition by using **quantifiers**.
Universal: $\forall x$ means for all/for every assigned value a of x.
Existential: $\exists x$ means that for some assigned values a of x.

Quantified statements

in symbols	in words
"$\forall x \in D, P(x)$." or "$(\forall x \in D)P(x)$."	"For every $x \in D$, $P(x)$." "If $x \in D$, then $P(x)$."
"$\exists x \in D \ni P(x)$" or "$(\exists x \in D)P(x)$"	"There exists x such that $P(x)$."

Once a quantifier is applied to a variable, then the variable is called a **bound** variable. The variable that is not bound is called a **free** variable.

1. The area of a rectangle is its length times its width.
   Quantifiers:

   ____

2. A triangle may be equilateral.
   Quantifiers:

   ____

3. $15 - 5 = 10$
   Quantifiers:

4. A real-valued function that is continuous at 0 is not necessarily differentiable at 0.
   Quantifiers:

   ____

EXAMPLE 19. *For a triangle T, let*
$P(T)$: *T is equilateral* $Q(T)$: *T is isosceles.*
State $\forall T, P(T) \Rightarrow Q(T)$ in a variety of ways completing the following.

1. *If T is an* ________________ *triangle, then T is* ________________.
2. *A triangle* ________________ *is isosceles, if T is* ________________.
3. *A triangle T being* ________________ *implies that T is* ________________.
4. *A triangle T is* ________________ *only if T is* ________________.
5. *For a triangle T to be* ________________, *it is* sufficient *that T is* ________________.
6. *For a triangle T to be* ________________, *it is* necessary *that T is* ________________.
7. *Every* ________________ *triangle is* ________________ .
8. *Whenever a triangle is* ________________ , *it is* ________________ .

EXAMPLE 20. *Consider the following predicates*
$P(x)$: *x is a multiple of 4.* $Q(x)$: *x is even. Complete:*

- *"For every integer integer $x, P(x) \Rightarrow Q(x)$" is* ______________________.
- *$P(x)$ is a* ______________________*condition for Q to be true.*
- *$Q(x)$ is a*______________________ *condition for $P(x)$ to be true.*
- *$Q(x)$ is not a* ______________________ *condition for $P(x)$ to be true.*

EXAMPLE 21. *Consider the following predicates*
$P(f)$: f is a differentiable function.
$Q(f)$: f is a continuous function.
Complete:

- *"For every real-valued function $f, P(f) \Rightarrow Q(f)$" is* ______________________.

- *"For every real-valued function $f, Q(f) \Rightarrow P(f)$" is* ______________________.
- *$Q(f)$ is a* ______________________ *condition for f to be differentiable, but not a* ______________________ *condition.*
- *$P(f)$ is a* ______________________ *condition for f to be continuous.*

EXAMPLE 22. *If m and n are odd integers then $m+n$ is even.*

Rewrite the statement in symbols. Then write its contrapositive and converse both in symbols and words.

EXAMPLE 23. *Rewrite the following statements in symbols using quantifiers. Introduce variables, where appropriate.*

a) *For every real number x, the formula $x+5=7$ holds.*

____

b) *All positive real numbers have a square root.*

____

c) *The sum of an even integer and an odd integer is even.*

____

d) *For every integer n, either $n \leq 1$ or $n^2 \geq 4$.*

____

NEGATIONS of quantified statements

1. All continuous functions are differentiable.

   ____

   ____

   ____

2. P :There exist real numbers a and b such that $(a+b)^2 = a^2 + b^2$.

   $\neg P$____

Quantified statement	Corresponding negation
$\forall x \in D, P(x)$	$\exists x \in D \ni (\neg P(x))$
$\exists x \in D \ni P(x)$	$\forall x \in D, (\neg P(x))$
$\forall x \in D, (P(x) \vee Q(x))$	
$\exists x \in D \ni (P(x) \wedge Q(x))$	
$\forall x \in D, (P(x) \Rightarrow Q(x))$	

EXAMPLE 24. *Negate the statements below using the following steps:*

1. *Rewrite P in symbols using quantifiers (it might be helpful to introduce temporary variables).*
2. *Express the negation of P in symbols using the above rules.*
3. *Express $\neg P$ in words (try to use the same wording as in original statement).*

a) P: *There exists a prime number p which is greater than 7 and less than 10.*

P ____

$\neg P$ ____

$\neg P$ ____

b) P : *For every even integer n there exists an integer m such that $n = 2m$.*

P ____

$\neg P$ ____

$\neg P$ ____

c) P : *If n is an odd integer then $3n + 7$ is odd.*

P ____

$\neg P$ ____

$\neg P$ ____

d) P: *If* n *is an integer and* n^2 *is a multiple of* 4 *then* n *is a multiple of* 4.

P ____

$\neg P$ ____

$\neg P$ ____

____

EXAMPLE 25. *Express the following statements in the form "for all ... , if ... then ..." using symbols to represent variables. Then write their negations in words, again using symbols for introduced variables.*

(a) S : *Every octagon has eight sides.*

(b) S : *Between any two real numbers there is a rational number.*

REMARK 26. Often, a more useful way to express the negated statement is to express it "positively", if possible. Using Trichotomy Axiom, DeMorgan's Law etc is useful. Also if the statement is quantified, the quantifier must be identified and replaced accordingly in order to get a useful denial.