5.5 FUNCTIONS AND SETS

Image of a Set

DEFINITION 1. *Let $f : X \to Y$ be a function. If $A \subseteq X$, we define $f(A)$, the* **image** *of A under f, by*

$$f(A) = \{y \in Y | y = f(x) \quad for \quad some \quad x \in A\}.$$

EXAMPLE 2. *Let $f : X \to Y$. Complete:*

(a) *If $A \subseteq X$ then $f(A) \subseteq$ ____ $\subseteq$ ____*

(b) $f(X) =$ ____

(c) $y \in f(A) \Leftrightarrow$ ______________________________

(d) $y \notin f(A) \Leftrightarrow$ ______________________________

EXAMPLE 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos x$. Find the following*

(a) $f([-\pi/2, \pi/2])$

(b) $f([-\pi/2, 0])$

(c) $f([-\pi/2, \pi/2017])$

(d) $f([-\pi/4, \pi/4])$

(e) *Does $-\dfrac{1}{2017}$ belong to $f([-\pi/2, \pi/2])$?*

Inverse Image

DEFINITION 4. *Let $f : X \to Y$ be a function and let B be a subset of its codomain (i.e. $B \subseteq Y$). Then the* **inverse image** *of B (written $f^{-1}(B)$) is the set*

$$f^{-1}(B) = \{x \in X | f(x) \in B\}.$$

The inverse image $f^{-1}(B)$ is a subset of domain of f containing all preimages of points from B under f.

EXAMPLE 5. *Let $f : X \to Y$ and $B \subseteq Y$. Complete:*

(a) $f^{-1}(B) \subseteq$ ________

(b) $x \in f^{-1}(B) \Leftrightarrow$ ___________________________

(c) $x \notin f^{-1}(B) \Leftrightarrow$ ___________________________

(d) *If $x \in X$ and $y \in Y$, then*

$$f(x)\_\_\_\_Y, \quad f(\{x\})\_\_\_\_Y, \quad , f^{-1}(y)\_\_\_\_X, \quad f^{-1}(\{y\})\_\_\_\_X,$$

EXAMPLE 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^4$. Find the following:*

(a) $f([1, 2]) =$

(b) $f([-2, -1]) =$

(c) $f^{-1}([1, 16]) =$

(d) $f^{-1}([-16, -1]) =$

(e) $f^{-1}([-1, 1]) =$

(f) $f^{-1}(\mathbb{R}) =$

(g) $f^{-1}(\mathbb{R}^+) =$

EXAMPLE 7. *Let* $A \subseteq \text{dom}(f)$ *and* $B \subset \text{codom}(f)$. *Determine the truth or falsehood of the following*

(a) $f^{-1}(f(A)) = A$

(b) $f(f^{-1}(B)) = B$

EXAMPLE 8. *Let* $A_1, A_2 \subseteq \text{dom}(f)$. *Determine the truth or falsehood of the following*

$$f(A_1) \cap f(A_2) = f(A_1 \cap A_2)$$

Summary

Let $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$. Then above definitions imply the following tautologies

- $\forall y \in Y, ((y \in \operatorname{ran}(f)) \Leftrightarrow (\exists x \in X \ni f(x) = y))$.
- $\forall y \in Y, ((y \in f(A)) \Leftrightarrow (\exists x \in A \ni f(x) = y))$.
- $\forall x \in X, \big((x \in f^{-1}(B)) \Leftrightarrow (f(x) \in B)\big)$.

Also note that

- If $B \subseteq \operatorname{ran}(f)$ then $(S = f^{-1}(B)) \Rightarrow (f(S) = B)$.

EXAMPLE 9. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 4$ and let $B = \{x \in \mathbb{R} | x > 0\}$. Find $f^{-1}(B)$. Justify your answer (give a formal proof).*

EXAMPLE 10. *$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 5x - 4$. Find $f([0, 1])$. Justify your answer (give a formal proof).*

PROPOSITION 11. *Let* $f : X \to Y$. *If* $A_1 \subseteq A_2 \subseteq X$ *then* $f(A_1) \subseteq f(A_2)$.

Proof.

PROPOSITION 12. *Let* $f : X \to Y$. *If* B_1 *and* B_2 *are subsets of* Y *then*

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2).$$

Proof.

EXAMPLE 13. *(cf. Example 8.) Let* $f : X \to Y$ *and* A_1 *and* A_2 *be subsets of* X.

(a) *Prove that* $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

(b) *Prove that if, in addition,* f *is an injective function, then* $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

EXAMPLE 14. *(cf. Example 7.) Let $f : X \to Y$ and B be a subset of Y.*

(a) *Prove that $f(f^{-1}(B)) \subseteq B$.*

(b) *Prove that if, in addition, f is an surjective function, then $f(f^{-1}(B) = B$.*