11.2: Vectors and the Dot Product in Three Dimensions

DEFINITION 1. A 3-dimensional vector is an ordered triple $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector **a** with representation \overrightarrow{PQ} is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$
.

The representation of the vector that starts at the point O(0,0,0) and ends at the point $P(x_1, y_1, z_1)$ is called the **position** vector of the point P.

EXAMPLE 2. Find the vector represented by the directed line segment with the initial point A(1,2,3) and terminal point B(3,2,-1). What is the position vector of the point A?

Vector Arithmetic: Let $a = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- Scalar Multiplication: $\alpha \mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle, \ \alpha \in \mathbb{R}.$
- Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

TRIANGLE LAW

PARALLELOGRAM LAW

Two vectors **a** and **b** are parallel if one is a scalar multiple of the other, i.e. there exists $\alpha \in \mathbb{R}$ s.t. $\mathbf{b} = \alpha \mathbf{a}$. Equivalently:

$$\mathbf{a} \| \mathbf{b} \iff$$

The magnitude or length of $a = \langle a_1, a_2, a_3 \rangle$:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Zero vector: $\mathbf{0} = \langle 0, 0, 0 \rangle, |0| = 0.$

Note that $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

Unit vector in the same direction as \mathbf{a} : $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$ The process of multiplying a vectoe \mathbf{a} by the reciprocal of its length to obtain a unit vector with the same direction is called normalizing \mathbf{a} .

Note that in \mathbb{R}^2 a nonzero vector **a** can be determined by its length and the angle from the positive x-axis:

In \mathbb{R}^2 and \mathbb{R}^3 a vector can be determined by its length and a vector in the same direction:

$$\mathbf{a} = |\mathbf{a}| \, \hat{\mathbf{a}},$$

i.e. a is equal to its length times a unit vector in the same direction.

EXAMPLE 3. Find the components of a vector \mathbf{a} of length $\sqrt{5}$ that extends along the line through the points M(2,5,0) and N(0,0,4).

Standard Basis Vectors:

$$\mathbf{i} = \langle 1, 0, 0 \rangle$$

$$\mathbf{j} = \langle 0, 1, 0 \rangle$$

$$\mathbf{k} = \langle 0, 0, 1 \rangle$$

Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.

We have:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle =$$

EXAMPLE 4. Given
$$\mathbf{a} = \langle 1, 0, -3 \rangle$$
 and $\mathbf{b} = \langle 3, 1, 2 \rangle$. Find

$$(a)\ |b-a|.$$

(b) a unit vector that has the same direction as b.

Dot Product of two nonzero vectors **a** and **b** is the NUMBER:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos \theta,$$

where θ is the angle between **a** and **b**, $0 \le \theta \le \pi$.

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = 0$.

Component Formula for dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

If θ is the *angle* between two nonzero vectors **a** and **b**, then

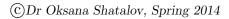
$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = -$$

DEFINITION 5. Two nonzero vectors **a** and **b** are called **perpendicular** or orthogonal if the angle between them is $\theta = \pi/2$.

EXAMPLE 6. For two nonzero vectors a and b show that

(a)
$$\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0$$

(b)
$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$



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 $EXAMPLE~7.~\textit{For what } value(s)~\textit{of } c~\textit{are the vectors } c\mathbf{i} + 2\mathbf{j} + \mathbf{k}~\textit{and } 4\mathbf{i} + 3\mathbf{j} + c\mathbf{k}~\textit{orthogonal?}$

 $EXAMPLE \ 8. \ \textit{The points } A(6,-1,0), \ B(-3,1,2), \ C(2,4,5) \ \textit{form a triangle. Find angle at } A.$