11.2: Vectors and the Dot Product in Three Dimensions

DEFINITION 1. *A* 3-**dimensional vector** *is an ordered triple* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{PQ}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

The representation of the vector that starts at the point $O(0, 0, 0)$ and ends at the point $P(x_1, y_1, z_1)$ is called the **position** vector of the point P.

EXAMPLE 2. *Find the vector represented by the directed line segment with the initial point* $A(1, 2, 3)$ *and terminal point* $B(3, 2, -1)$. *What is the position vector of the point* A*?*

Vector Arithmetic: Let $a = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- Scalar Multiplication: $\alpha\mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$, $\alpha \in \mathbb{R}$.
- Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

TRIANGLE LAW PARALLELOGRAM LAW

Two vectors **a** and **b** are parallel if one is a scalar multiple of the other, i.e. there exists $\alpha \in \mathbb{R}$ s.t. $\mathbf{b} = \alpha\mathbf{a}$. Equivalently:

$$\mathbf{a} \| \mathbf{b} \quad \Leftrightarrow$$

The magnitude or length of $a = \langle a_1, a_2, a_3 \rangle$:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

`Zero vector:` $\mathbf{0} = \langle 0, 0, 0 \rangle$, $|0| = 0$.
Note that $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

`Unit vector in the same direction as a:` $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ *The process of multiplying a vectoe* **a** *by the reciprocal of its length to obtain a unit vector with the same direction is called* **normalizing a**.

Note that in $\mathbb{R}^2$ a nonzero vector **a** can be determined by its length and the angle from the positive x-axis:

In $\mathbb{R}^2$ and $\mathbb{R}^3$ a vector can be determined by its length and a vector in the same direction:

$$\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{a}},$$

i.e. **a** is equal to its length times a unit vector in the same direction.

EXAMPLE 3. *Find the components of a vector* **a** *of length* $\sqrt{5}$ *that extends along the line through the points* $M(2, 5, 0)$ *and* $N(0, 0, 4)$.

Standard Basis Vectors:
$\mathbf{i} = \langle 1, 0, 0 \rangle$
$\mathbf{j} = \langle 0, 1, 0 \rangle$
$\mathbf{k} = \langle 0, 0, 1 \rangle$
Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.
We have:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle =$$

EXAMPLE 4. *Given* $\mathbf{a} = \langle 1, 0, -3 \rangle$ *and* $\mathbf{b} = \langle 3, 1, 2 \rangle$. *Find*

(a) $|\mathbf{b} - \mathbf{a}|$.

(b) *a unit vector that has the same direction as* $\mathbf{b}$.

Dot Product of two nonzero vectors **a** and **b** is the NUMBER:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\theta,$$

where θ is the angle between **a** and **b**, $0 \le \theta \le \pi$.

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = 0$.

Component Formula for dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

If θ is the *angle* between two nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \underline{\hspace{8cm}}$$

DEFINITION 5. *Two nonzero vectors* **a** *and* **b** *are called* **perpendicular** *or orthogonal if the angle between them is* $\theta = \pi/2$.

EXAMPLE 6. *For two nonzero vectors* **a** *and* **b** *show that*

(a)

$$\mathbf{a} \perp \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \cdot \mathbf{b} = 0$$

(b)

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

EXAMPLE 7. *For what value(s) of* c *are the vectors* $c\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ *and* $4\mathbf{i} + 3\mathbf{j} + c\mathbf{k}$ *orthogonal?*

EXAMPLE 8. *The points* $A(6, -1, 0)$, $B(-3, 1, 2)$, $C(2, 4, 5)$ *form a triangle. Find angle at* A.