11.6: Vector Functions and Space Curves

A vector function is a function that takes one or more variables and returns a vector. Let $\mathbf{r}(t)$ be a vector function whose range is a set of 3-dimensional vectors:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

where $x(t), y(t), z(t)$ are functions of one variable and they are called the **component functions**.

A vector function $\mathbf{r}(t)$ is *continuous* if and only if its component functions $x(t), y(t), z(t)$ are continuous.

EXAMPLE 1. *Given*

$$\mathbf{r}(t) = \left\langle t\ln(t+1), t^2\sin t, e^t\right\rangle.$$

(a) *Find the domain of* $\mathbf{r}(t)$.

(b) *Find all* t *where* $\mathbf{r}(t)$ *is continuous.*

Space curve is given by **parametric equations***:*

$$C = \{(x, y, z) | x = x(t), y = y(t), z = z(t), \quad t \text{ in } I\},$$

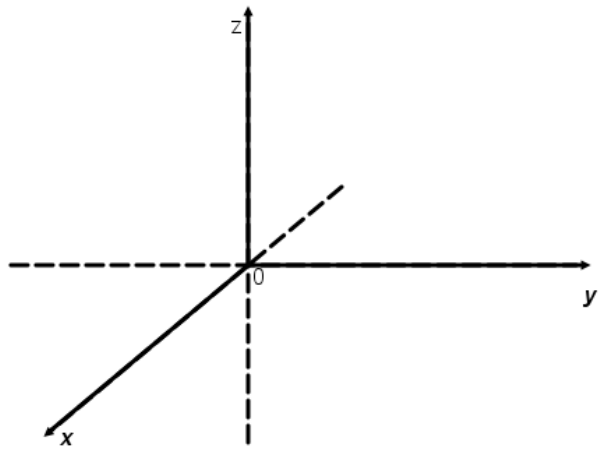
where I *is an interval and* t *is a* **parameter***.*

FACT: Any continuous vector-function $\mathbf{r}(t)$ *defines a space curve* C *that is traced out by the tip of the moving vector* $\mathbf{r}(t)$*.*

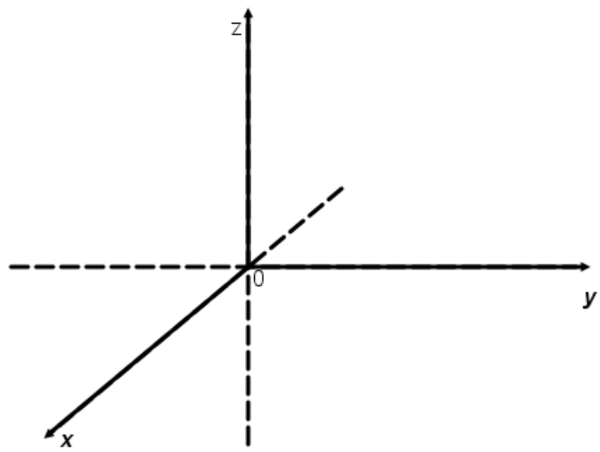
Any parametric curve has a **direction of motion** *given by increasing of parameter.*

EXAMPLE 2. *Describe the curve defined by the vector function (indicate direction of motion):*

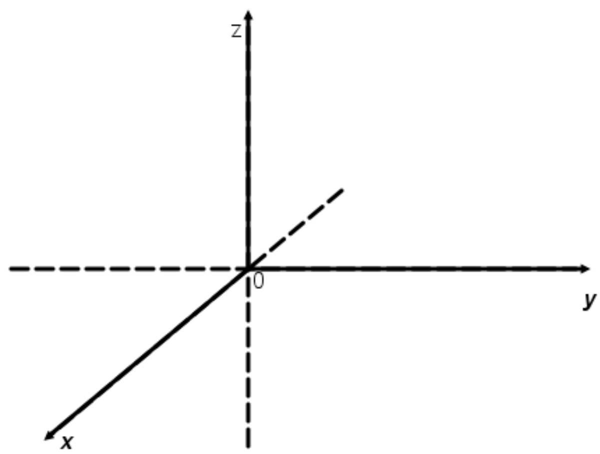
(a) $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$



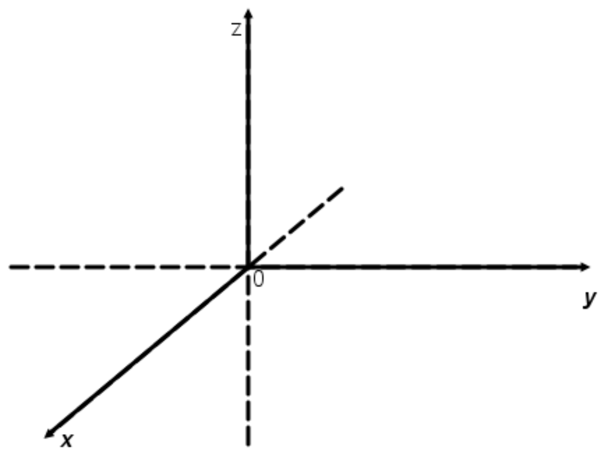
(b) $\mathbf{r}(t) = \langle \cos at, \sin at, c \rangle$ *where* a *and* c *are positive constants.*



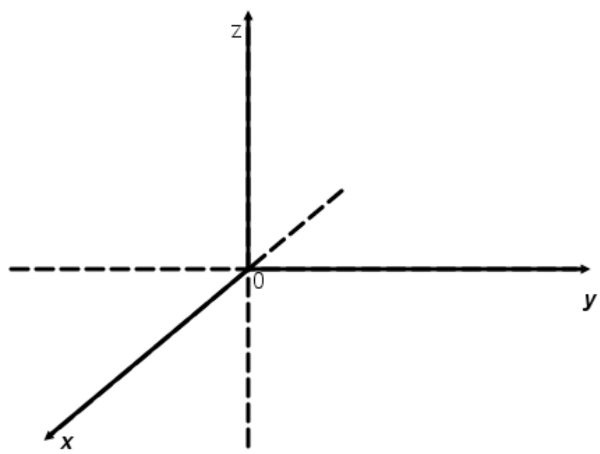
(c) $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 1 \rangle$, $0 \le t \le 2\pi$



(d) $\mathbf{r}(t)(x) = \langle \cos t, \sin t, t \rangle$



(e) $\mathbf{r}(t) = \langle 1 + t, 3 + 2t, 4 - 5t \rangle$, $0 \le t \le -1$.



EXAMPLE 3. *Show that the the curve given by*

$$\mathbf{r}(t) = \left\langle \sin t, 2\cos t, \sqrt{3}\sin t \right\rangle$$

lies on both a plane and a sphere. Then conclude that its graph is a circle and find its radius.

Derivatives: *The derivative* $\mathbf{r}'$ *of a vector function* $\mathbf{r}$ *is defined just as for a real-valued function:*

$$\frac{\mathrm{d}\mathbf{r}(t_0)}{\mathrm{d}t} = \mathbf{r}'(t_0) = \lim_{h\to 0} \frac{\mathbf{r}(t_0+h) - \mathbf{r}(t_0)}{h}$$

if the limit exists. The derivative $\mathbf{r}'(t_0)$ *is the* tangent vector *to the curve* $\mathbf{r}(t)$ *at the point* $\mathbf{r}(t_0) = \langle x(t_0), y(t_0), z(t_0)\rangle$.

THEOREM 4. *If the functions* $x(t), y(t), z(t)$ *are differentiable, then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

EXAMPLE 5. *Given* $\mathbf{r}(t) = (1+t)^2\mathbf{i} + e^t\mathbf{j} + \sin 3t\mathbf{k}$.

(a) *Find* $\mathbf{r}'(t)$

(b) *Find a tangent vector to the curve at* $t = 0$.

(c) *Find a tangent line to the curve at* $t = 0$.

(c′) *Find a tangent line to the curve at the point* $< 1, 1, 0 >$.