

12.1: Functions of Several Variables

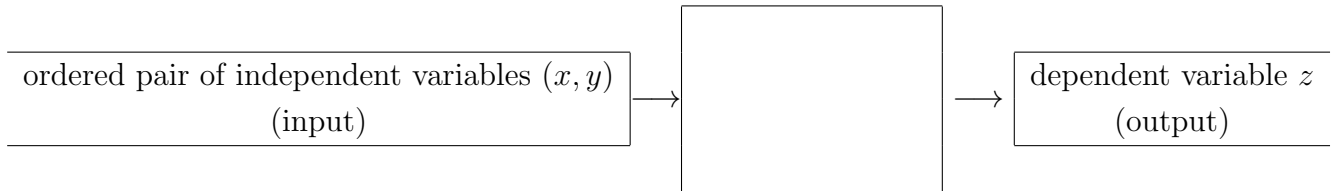
Consider the following formulas:

$$z = 2 - x - 4y \quad (1)$$

$$z = x^2 + y^2 \quad (2)$$

$$z = \sqrt{x^2 + y^2} \quad (3)$$

$$z = \sqrt{1 - x^2 - y^2} \quad (4)$$



DEFINITION 1. Let $D \subset \mathbb{R}^2$. A **function f of two variables** is a rule that assigns to each ordered pair (x, y) in D a unique real number denoted by $f(x, y)$.

The set D is the **domain** of f and the **range** of f is the set of values that f takes on, that is $\{f(x, y) | (x, y) \in D\}$.

REMARK 2. Obviously, one can choose the independent variables arbitrary, for example, $x = f(y, z)$.

- **GRAPH** of $f(x, y)$.

Recall that a graph of a function f of one variable is a curve C with equation $y = f(x)$.

DEFINITION 3. The **graph** of f with domain D is the set:

$$S = \{(x, y, z) \in \mathbb{R}^3 | z = f(x, y), (x, y) \in D\}.$$

The graph of a function f of two variables is a surface S in three dimensional space with equation $z = f(x, y)$.

EXAMPLE 4. Find the domain and sketch the graph of the functions (1)-(4). What is the range?

$$(1) z = 2 - x - 4y$$

$$D =$$

$$(2) z = x^2 + y^2$$

$$D =$$

$$(3) z = \sqrt{x^2 + y^2}$$

$$D =$$

$$(4) z = \sqrt{1 - x^2 - y^2}$$

$$D =$$

EXAMPLE 5. Sketch the domain of each of the following:

(a) $z = \sqrt{x} - \frac{5}{\sqrt{y}}$



(b) $z = \ln(x^2 + \frac{y^2}{16} - 1)$



• **LEVEL (CONTOUR) CURVES** method of visualizing functions is the method borrowed from mapmakers. It is a contour map on which points of constant elevation are joined to form level (or contour) curves.

DEFINITION 6. The **level (contour) curves** of a function of two variables are the curves with equations

$$f(x, y) = k,$$

where k is a constant in the range of f .

A level curve is the locus of all points at which f takes a given value k (it shows where the graph of f has height k).

EXAMPLE 7. Sketch the level curves of the functions (2) and (3) for the values $k = 0, 1, 2, 3, 4$:

$$(2) z = x^2 + y^2$$

$$(3) z = \sqrt{x^2 + y^2}$$

- **Functions of three variables.**

DEFINITION 8. Let $D \subset \mathbb{R}^3$. A **function f of three variables** is a rule that assigns to each ordered pair (x, y, z) in D a unique real number denoted by $f(x, y, z)$.

Examples of functions of 3 variables:

$$f(x, y, z) = x^2 + y^2 + z^2,$$

$$u = xyz$$

$$T(s_1, s_2, s_3) = \ln s_1 + 12s_2 - s_3^{-5}.$$

Emphasize that domains of functions of three variables are regions in three dimensional space.

EXAMPLE 9. Find the domain of the following function:

$$f(x, y, z) = \frac{\ln(36 - x^2 - y^2 - z^2)}{\sqrt{x^2 + y^2 + z^2 - 25}}.$$

Note that for functions of three variables it is impossible to visualize its graph. However we can examine them by their **level surfaces**:

$$f(x, y, z) = k$$

where k is a constant in the range of f . If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

EXAMPLE 10. Find the level surfaces of the function $u = x^2 + y^2 - z$.

REMARK 11. For any function there exist a unique level curve (surface) through given point!!!