12.3: Partial Derivatives

DEFINITION 1. *If f is a function of two variables, its* **partial derivatives** *are the functions f_x and f_y defined by*

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Conclusion: $f_x(x,y)$ represents the *rate of change* of the function $f(x,y)$ as we change x and hold y fixed while $f_y(x,y)$ represents the rate of change of $f(x,y)$ as we change y and hold x fixed.

Notations for partial derivatives: If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$
$$f_y(x,y) = f_y =$$

RULE FOR FINDING PARTIAL DERIVATIVES OF $z = f(x,y)$:

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

EXAMPLE 2. *If $f(x,y) = x^3 + y^5 e^x$ find $f_x(0,1)$ and $f_y(0,1)$.*

EXAMPLE 3. *Find all of the first order partial derivatives for the following functions:*

(a) $z(x,y) = x^3 \sin(xy)$

(b) $u(x, y, z) = ye^{xyz}$

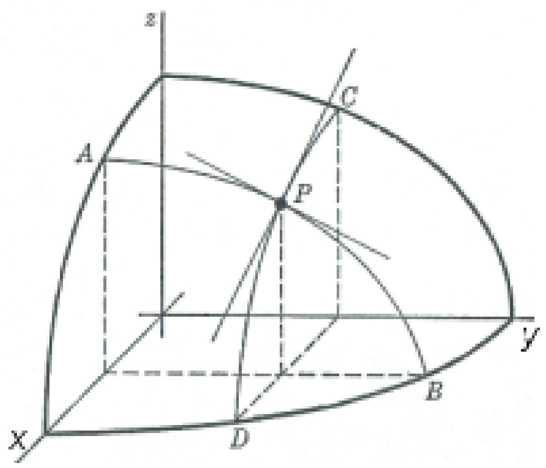
EXAMPLE 4. *The temperature at a point* (x, y) *on a flat metal plate is given by*

$$T(x, y) = \frac{80}{1 + x^2 + y^2},$$

where T *is measured in* °C *and* x, y *in meters. Find the rate of change of temperature with respect to distance at the point* $(1, 2)$ *in the* y*-direction.*

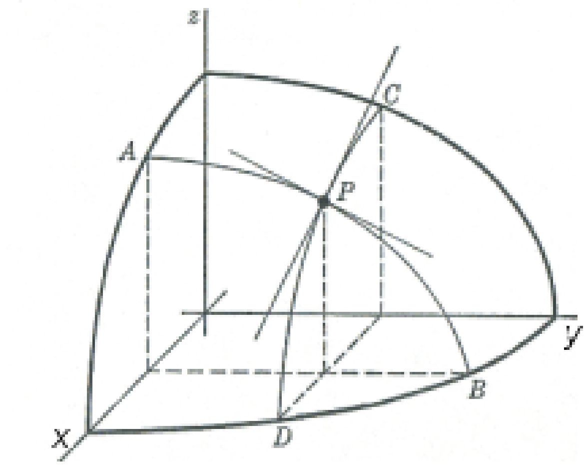
Geometric interpretation of partial derivatives: Partial derivatives are the *slopes of traces*:

- $f_x(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $y = b$ at the point (a,b).



- $f_y(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $x = a$ at (a,b).

EXAMPLE 5. *If* $f(x,y) = \sqrt{4 - x^2 - 4y^2}$, *find* $f_x(1,0)$ *and* $f_y(1,0)$ *and interpret these numbers as slopes. Illustrate with sketches.*



Higher derivatives: Since both of the first order partial derivatives for $f(x, y)$ are also functions of x and y, so we can in turn differentiate each with respect to x or y. We use the following notation:

$$\begin{array}{lllllll}
(f_x)_x &= f_{xx} &= f_{11} &= \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) &= \dfrac{\partial^2 f}{\partial x^2} &= \dfrac{\partial^2 z}{\partial x^2} \\
(f_x)_y &= f_{xy} &= f_{12} &= \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) &= \dfrac{\partial^2 f}{\partial y \partial x} &= \dfrac{\partial^2 z}{\partial y \partial x} \\
(f_y)_x &= f_{yx} &= f_{21} &= \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) &= \dfrac{\partial^2 f}{\partial x \partial y} &= \dfrac{\partial^2 z}{\partial x \partial y} \\
 & & & & & \\
(f_y)_y &= &= &= &= &=
\end{array}$$

EXAMPLE 6. *Find the second partial derivatives of*

$$f(x, y) = y^3 + 5y^2 e^{4x} - \cos(x^2).$$

Clairaut's Theorem. *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Partial derivative of order three or higher can also be defined. For instance,

$$f_{yyx} = (f_{yy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial x \partial y^2}.$$

Using Clairaut's Theorem one can show that if the functions f_{yyx}, f_{xyy} and f_{yxy} are continuous then

EXAMPLE 7. *Find the indicated derivative for*

$$f(x, y, z) = \cos(xy + z).$$

(a) f_{xy}

(b) f_{zxy}