

12.5: The Chain Rule

Chain Rule for functions of a single variable: If $y = f(x)$ and $x = g(t)$ where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}.$$

EXAMPLE 1. Let $z = x^y$, where $x = t^2$, $y = \sin t$. Compute $z'(t)$.

Assume that all functions below have continuous derivatives (ordinary or partial).

- CASE 1: $z = f(x, y)$, where $x = x(t)$, $y = y(t)$ and compute $z'(t)$.

Chain Rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

SOLUTION OF EXAMPLE 1:

EXAMPLE 2. The radius of a right circular cone is increasing at a rate of 1.8 cm/s while its height is decreasing at a rate 2.5 cm/s. At what rate is the volume of the cone changing when the radius is 120 cm and the height is 140 cm.

- CASE 2: $z = f(x, y)$, where $x = x(s, t)$, $y = y(s, t)$ and compute z_s and z_t .

Chain Rule:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

Tree diagram:

EXAMPLE 3. Write out the Chain Rule for the case where $w = f(x, y, z)$ and $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$.

EXAMPLE 4. If $z = \sin x \cos y$, where $x = (s - t)^2$, $y = s^2 - t^2$ find $z_s + z_t$.

EXAMPLE 5. If $u = x^2y + y^3z^2$ where $x = rse^t$, $y = r + s^2e^{-t}$, $z = rs \sin t$, find u_s when $(r, s, t) = (1, 2, 0)$

Implicit differentiation: Suppose that an equation

$$F(x, y) = 0$$

defines y implicitly as a differentiable function of x , i.e. $y = y(x)$, where $F(x, y(x)) = 0$ for all x in the domain of $y(x)$. Find y' :

EXAMPLE 6. Find y' if $x^4 + y^3 = 6e^{xy}$.

Suppose that an equation

$$F(x, y, z) = 0$$

defines z implicitly as a differentiable function of x and y , i.e. $z = z(x, y)$, where

$$F(x, y, z(x, y)) = 0$$

for all (x, y) in the domain of z . Find the partial derivatives z_x and z_y :

EXAMPLE 7. If $x^4 + y^3 + z^2 + xye^z = 10$ find

(a) z_x and z_y

(b) x_y and x_z