12.6: Directional Derivatives and the Gradient Vector

Recall that the two partial derivatives $f_x(x,y)$ and $f_y(x,y)$ of $f(x,y)$ represent the rate of change of f as we vary x (holding y fixed) and as we vary y (holding x fixed) respectively. In other words, $f_x(x,y)$ and $f_y(x,y)$ represent the rate of change of f in the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ respectively. Let's consider how to find the rate of change of f if we allow both x and y to change simultaneously, or how to find the rate of change of f in the direction of an arbitrary vector $\mathbf{u}$.

DEFINITION 1. *The rate of change of $f(x,y)$ in the direction of the unit vector $\hat{\mathbf{u}} = \langle a, b\rangle$ is called the* **directional derivative** *and it is denoted by $D_{\mathbf{u}}f(x,y)$.*

The **directional derivative** *of f at (x_0, y_0) in the direction of the unit vector $\hat{\mathbf{u}} = \langle a, b\rangle$ is*

$$D_{\mathbf{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

if this limit exists.

REMARK 2. By comparing the last definition with the definitions of the partial derivatives:

$$f_x(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}, \quad f_y(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

we see that

$f_x(x_0,y_0) =$ and $f_y(x_0,y_0) =$

For computational purposes use the following theorem.

THEOREM 3. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\hat{\mathbf{u}} = \langle a, b\rangle$ and*

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b.$$

EXAMPLE 4. *Find the rate of change $f(x,y) = x^3 + \sin(xy)$ at the point $(1, \pi/2)$ in the direction indicated by the angle $\theta = \pi/4$.*

The Directional Derivative As The Dot Product Of Two Vectors. Gradient.

DEFINITION 5. *The* **gradient** *of* $f(x,y)$ *is the vector function* ∇f *defined by*

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Notations for gradient: $\mathbf{grad} f$ or ∇f which is read "del f".

EXAMPLE 6. *Find the gradient of* $f = \cos(xy) + e^x$ *at (0,3).*

By Theorem 3 we have:

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b =$$

`Formula for the directional derivative using the gradient vector:`

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \hat{\mathbf{u}}.$$

EXAMPLE 7. *Find the directional derivative for* f *from Example 6 at* $(0,3)$ *in the direction* $\langle 3,4\rangle$.

The directional derivative of function of *three* variables

THEOREM 8. *If* f *is a differentiable function of* x, y *and* z, *then* f *has a directional derivative in the direction of any unit vector* $\hat{\mathbf{u}} = \langle a,b,c\rangle$ *and*

$$D_{\mathbf{u}} f(x,y) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c = \nabla f \cdot \hat{\mathbf{u}},$$

where

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is the gradient vector of $f(x,y,z)$.

EXAMPLE 9. *Find the directional derivative of* $f(x, y, z) = z^3 - x^2y$ *at the point* $(1, 6, 2)$ *in the direction* $\mathbf{u} = \langle 1, -2, 3 \rangle$.

QUESTION: In which of all possible directions does f *change fastest and what is the maximum rate of change.*

ANSWER is provided by the following theorem:

THEOREM 10. *Suppose* f *is a differentiable function of two or three variables. The maximum value of the directional derivative* $D_{\mathbf{u}}f$ *is* $|\nabla f|$ *and it occurs when* $\mathbf{u}$ *has the same direction as the gradient vector* ∇f.

Proof.

EXAMPLE 11. *Suppose that the temperature at a point (x, y, z) in the space is given by*

$$T(x, y, z) = \frac{100}{1 + x^2 + y^2 + z^2},$$

where T is measured in $^\circ C$ and x, y, z in meters.

(a) *In which direction does the temperature increase fastest at the point $(1, 1, -1)$?*

(b) *What is the maximum rate of increase?*

Tangent planes to level surfaces:

FACT: The gradient vector $\nabla F(x_0, y_0, z_0)$ is **orthogonal** to the level surface $F(x, y, z) = k$ at the point (x_0, y_0, z_0).

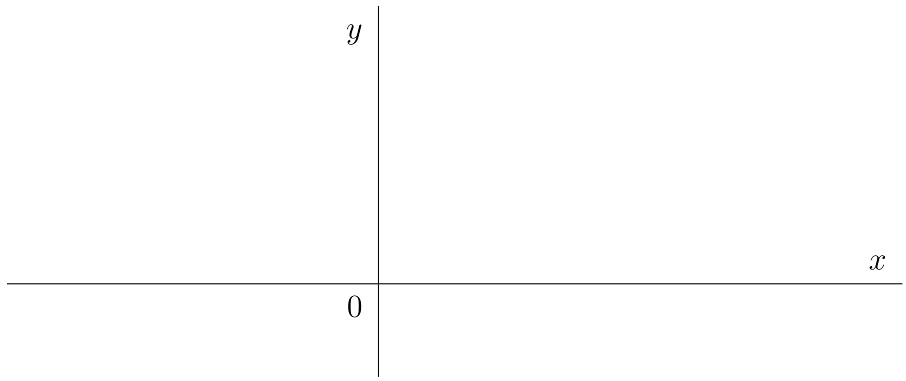
So, the *tangent plane* to the surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0) has the equation:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The normal line to the surface at the point (x_0, y_0, z_0) is the line passing through (x_0, y_0, z_0) and perpendicular to the tangent plane. Therefore its direction is given by the ______________ vector

EXAMPLE 12. *Find the equation of the tangent plane and normal line at the point $(1, 0, 5)$ to the surface $xe^{yz} = 1$.*

Likewise, the gradient vector $\nabla f(x_0, y_0)$ is **orthogonal** to the level curve $f(x, y) = k$ at the point (x_0, y_0).



Consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y). Draw a curve of steepest ascent.