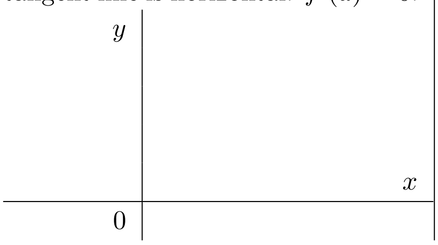
12.7: Maximum and minimum values

Function $y = f(x)$	Function of two variables $z = f(x, y)$
DEFINITION 1. *A function $f(x)$ has a local maximum at $x = a$ if $f(a) \geq f(x)$ when x is near a (i.e. in a neighborhood of a). A function f has a local minimum at $x = a$ if $f(a) \leq f(x)$ when x is near a.* .	DEFINITION 2. *A function $f(x, y)$ has a local maximum at $(x, y) = (a, b)$ if $f(a, b) \geq f(x, y)$ when (x, y) is near (a, b) (i.e. in a neighborhood of (a, b)). A function f has a local minimum at $(x, y) = (a, b)$ if $f(a, b) \leq f(x, y)$ when (x, y) is near (a, b).*
If the inequalities in this definition hold for ALL points x in the domain of f, then f has an **absolute max** (or **absolute min**) at a	If the inequalities in this definition hold for ALL points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).
If the graph of f has a tangent line at a local extremum, then the tangent line is horizontal: $f'(a) = 0$. 	If the graph of f has a tangent plane at a local extremum, then the tangent PLANE is horizontal.

THEOREM 3. *If f has a local extremum (that is, a local maximum or minimum) at (a, b) and the first-order partial derivatives exist there, then*

$$f_x(a, b) = f_y(a, b) = 0 \quad (\textit{or, equivalently,} \nabla f(a, b) = 0.)$$

DEFINITION 4. *A point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or one of this partial derivatives does not exist, is called a* **critical point** *of f.*

At a critical point, a function could have a local max or a local min, or neither.
We will be concerned with two important questions:

- Are there any local or absolute extrema?
- If so, where are they located?

SETS in $\mathbb{R}^2$

in $\mathbb{R}$	in $\mathbb{R}^2$
close interval $[a, b]$	close set
open interval (a, b)	open set
end points of an interval	boundary points

DEFINITION 5. *A* **bounded set** *in* $\mathbb{R}^2$ *is one that contained in some disk.*

THE EXTREME VALUE THEOREM:

Function $y = f(x)$	Function of two variables $z = f(x, y)$
If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(x_1)$ and an absolute minimum value $f(x_2)$ at some points x_1 and x_2 in $[a, b]$.	If f is continuous on a closed bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

EXAMPLE 6. *Find extreme values of* $f(x, y) = x^2 + y^2$.

	Local	Absolute
Maximum		
Minimum		

Domain:

EXAMPLE 7. *Find extreme values of* $f(x, y) = \sqrt{1 - x^2 - y^2}$.

	Local	Absolute
Maximum		
Minimum		

Domain:

EXAMPLE 8. *Find extreme values of* $f(x, y) = y^2 - x^2$.

	Local	Absolute
Maximum		
Minimum		

Domain:

REMARK 9. Example 8 illustrates so called **saddle point** of f. Note that the graph of f crosses its tangent plane at (a, b).

EXAMPLE 10. *Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.*

ABSOLUTE MAXIMUM AND MINIMUM VALUES on a closed bounded set.

THE EXTREME VALUE THEOREM:

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:	*To find the absolute max and min values of a continuous function f on a closed bounded set D:*
1. Find the values of f at the critical points of f in (a, b).	1. Find the values of f at the critical points of f in D.
2.Find the extreme values of f at the endpoints of the interval.	2.Find the extreme values of f on the boundary of D.(This usually involves the Calculus I approach for this work.)
3.The largest of the values from steps 1&2 is the absolute max value; the smallest of the values from steps 1&2 is the absolute min value.	3. The largest of the values from steps 1&2 is the absolute maximum value; the smallest of the values from steps 1&2 is the absolute minimum value.

- The quantity to me maximized/minimized is expressed in terms of variables (as few as possible!)
- Any constraints that are presented in the problem are used to reduce the number of variables to the point they are independent,
- After computing partial derivatives and setting them equal to zero you get purely algebraic problem (but it may be hard.)
- Sort out extreme values to answer the original question.

EXAMPLE 11. *A lamina occupies the region* $D = \{(x, y) : \ 0 \le x \le 3, \ \ -2 \le y \le 4 - 2x\}$. *The temperature at each point of the lamina is given by*

$$T(x, y) = 4(x^2 + xy + 2y^2 - 3x + 2y) + 10.$$

Find the hottest and coldest points of the lamina.

Second derivatives test:

Suppose f'' is continuous near a and $f'(c) = 0$ (i.e. a is a critical point).	Suppose that the second partial derivatives of f are continuous near (a, b) and $\nabla f(a, b) = \mathbf{0}$ (i.e. (a, b) is a critical point). Let $\mathcal{D} = \mathcal{D}(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$
• If $f''(c) > 0$ then $f(c)$ is a local minimum.	• If $\mathcal{D} > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
• If $f''(c) < 0$ then $f(c)$ is a local maximum.	• If $\mathcal{D} > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
	• If $\mathcal{D} < 0$ then $f(a, b)$ is not a local extremum (saddle point).

NOTE:

• If $f''(c) = 0$, then the test gives no information.	If $\mathcal{D} = 0$ or does not exist, then the test gives no information. fails.
	To remember formula for $\mathcal{D}$: $$\mathcal{D} = f_{xx}f_{yy} - [f_{xy}]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

EXAMPLE 12. *Use the Second Derivative Test to confirm that a local cold point of the lamina in the previous Example is* $(2, -1)$.

EXAMPLE 13. *Find the local extrema of* $f(x, y) = x^3 + y^3 - 3xy$.

Solution: Find critical points:

Calculate the second partial derivatives and $\mathcal{D}$.

$f_{xx} =$		
$f_{xy} =$		
$f_{yy} =$		
$\mathcal{D}$		

EXAMPLE 14. *The mountain is defined by $z = xy$ in the elliptical domain*

$$D = \left\{ (x, y) | \; \frac{x^2}{16} + y^2 \leq 1 \right\}.$$

(a) *Find the top of the mountain.*

(b) *Is the critical point found in the previous item the highest or the lowest in its neighborhood?*