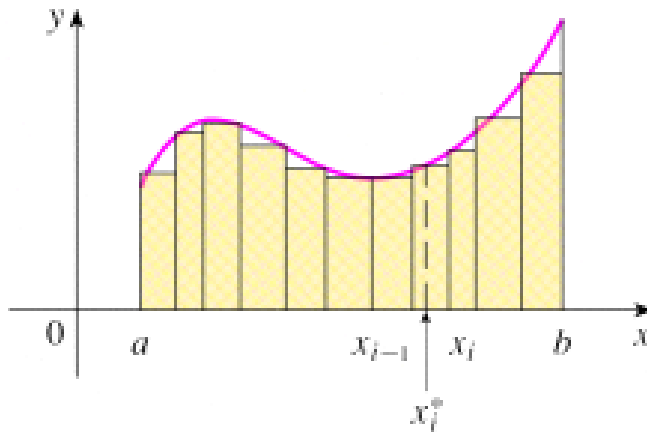


13.1: Double integrals over rectangles

Recall that a single definite integral can be interpreted as **area**:



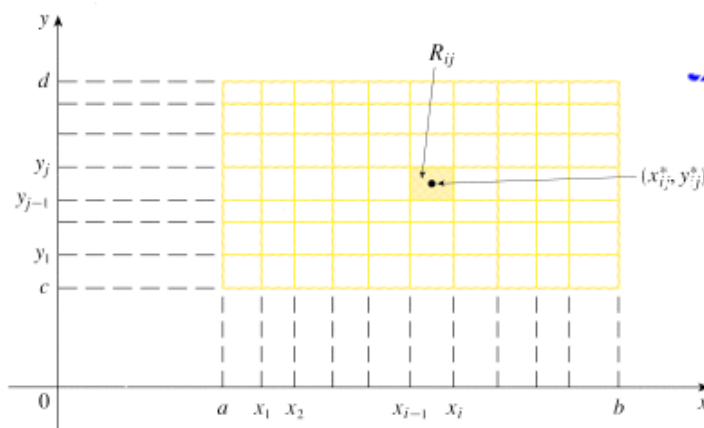
The exact area is also the definition of the definite integral:

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

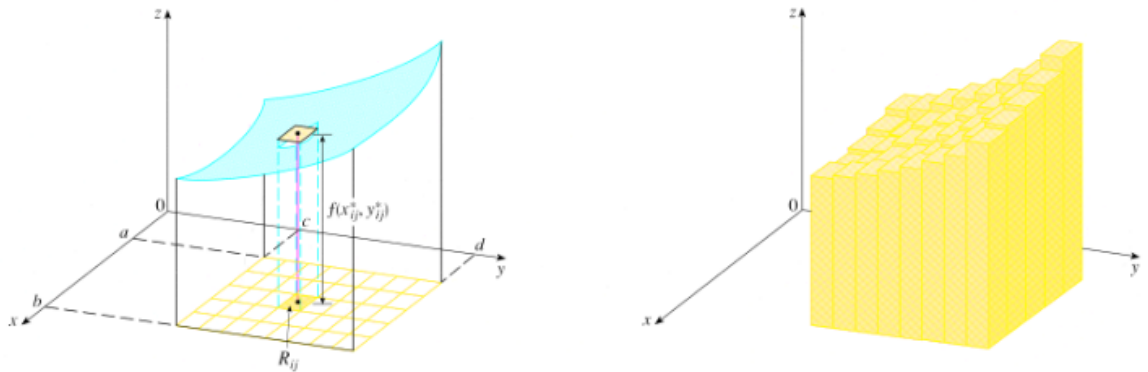
Problem: Assume that $f(x, y)$ is defined on a closed rectangle

$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$ and $f(x, y) \geq 0$ over R . Denote by S the part of the surface $z = f(x, y)$ over the rectangle R . What the volume of the region under S and above the xy -plane is?

Solution: Approximate the volume. Divide up $a \leq x \leq b$ into n subintervals and divide up $c \leq y \leq d$ into m subintervals. From each of these smaller rectangles choose a point (x_i^*, y_j^*) .



Over each of these smaller rectangles we will construct a box whose height is given by $f(x_i^*, y_j^*)$.



The volume is given by

$$\lim_{n,m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta x \Delta y$$

which is also the definition of a double integral

$$\iint_R f(x, y) dA.$$

Another notation: $\iint_R f(x, y) dA = \iint_R f(x, y) dx dy$.

THEOREM 1. *If f is continuous on R then f is integrable over R .*

THEOREM 2. *If $f(x, y) \geq 0$ and f is continuous on the rectangle $R = [a, b] \times [c, d]$, then the volume V of the solid S that lies above R and under the graph of f , i.e.*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in R, 0 \leq z \leq f(x, y), (x, y) \in R\},$$

is

$$V = \iint_R f(x, y) dA.$$

EXAMPLE 3. *Evaluate the integral*

$$\iint_R 4 dA$$

where $R = [-1, 0] \times [-3, 3]$ by identifying it as a volume of a solid.