

13.2: Iterated integrals

Suppose that $f(x, y)$ is integrable over the rectangle $R = [a, b] \times [c, d]$.

Partial integration of f with respect to x : $\int_a^b f(x, y) dx$

Partial integration of f with respect to y : $\int_c^d f(x, y) dy$

EXAMPLE 1.

$$\int_0^4 (x + 3y^2) dx =$$

$$\int_1^4 e^{xy} dy =$$

Iterated integrals:

$$\int_a^b \left[\int_c^d f(x, y) dy \right] dx = \int_a^b \int_c^d f(x, y) dy dx$$

and

$$\int_c^d \left[\int_a^b f(x, y) dx \right] dy = \int_c^d \int_a^b f(x, y) dx dy.$$

EXAMPLE 2. *Evaluate the integrals:*

$$I_1 = \int_0^1 \int_1^4 x\sqrt{y} dy dx, \quad I_2 = \int_1^4 \int_0^1 x\sqrt{y} dx dy$$

FUBINI's THEOREM: *If f is continuous on the rectangle*

$$R = [a, b] \times [c, d]$$

then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.$$

EXAMPLE 3. *Evaluate*

$$\iint_R x \cos(xy) \, dA$$

where $R = [-\pi/2, \pi/2] \times [1, 5]$

EXAMPLE 4. Find the volume of the solid S lying under the circular paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2, 2] \times [-3, 3]$.

FACT: If g and h are continuous functions of one variable and $R = [a, b] \times [c, d]$ then

$$\iint_R g(x)h(y) \, dA = \left(\int_a^b g(x) \, dx \right) \left(\int_c^d h(y) \, dy \right).$$

EXAMPLE 5. If $R = [0, \ln 2] \times [0, \ln 5]$ find $\iint_R e^{2x-y} \, dA$.