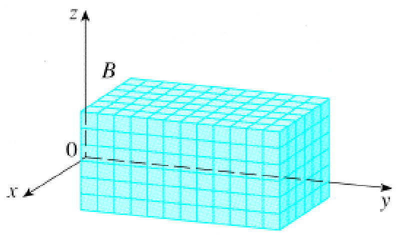
13.8: Triple Integrals

Mass problem: *Given a solid object, that occupies the region B in $\mathbb{R}^3$, with density $\rho(x, y, z)$. Find the mass of the object.*

Solution: Let B be a rectangular box:

$$B = \{(x, y, z)|a \le x \le b, c \le y \le d, r \le z \le s\}$$



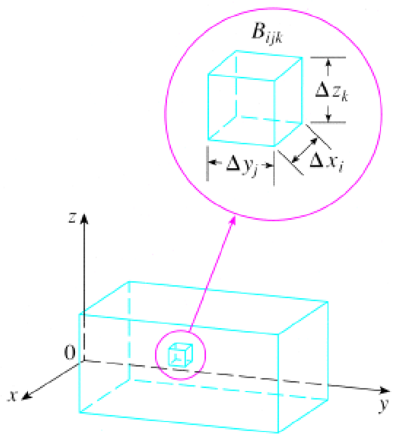
Partition in sub-boxes:

$$m_{ijk} = \rho(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

$$\|P\| = \max \sqrt{\Delta x_i^2 + \Delta y_j^2 + \Delta z_k^2}$$

$$m = \lim_{\|P\|\to 0} \sum_i \sum_j \sum_k \rho(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

$$m = \iiint_B \rho(x, y, z)\, \mathrm{d}V$$



[1]

FUBINI's THEOREM: *If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$ then*

$$\iiint_B f(x, y, z)\, \mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\mathrm{d}z$$

and there are 5 other possible orders in which we can integrate.

EXAMPLE 1. *Let $B = [0, 1] \times [-1, 3] \times [0, 3]$. Evaluate*

$$I = \iiint_B xye^{yz}\, \mathrm{d}V$$

[1] All figures are from the course textbook

FACT: The volume of the solid E is given by the integral,

$$V = \iiint_E \mathrm{d}V.$$

FACT: The mass of the solid E with variable density $\rho(x, y, z)$ is given by the integral,

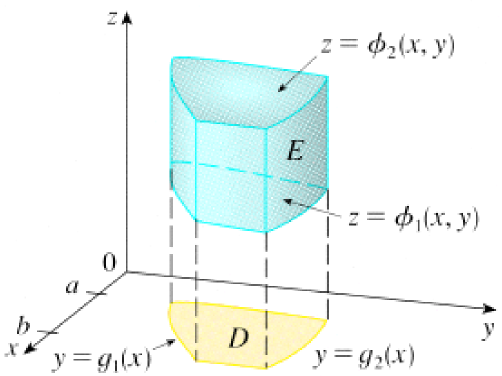
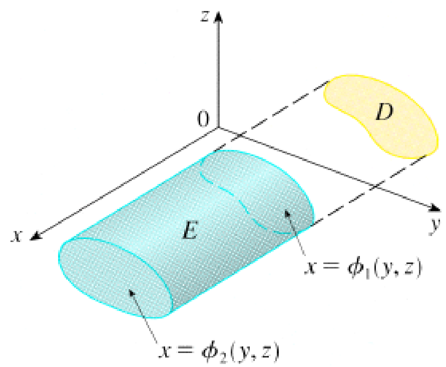
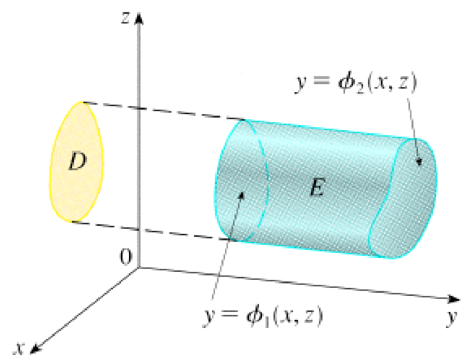
$$m = \qquad\qquad\qquad\qquad .$$

EXAMPLE 2. *Find the mass of the solid bounded by* $x = y^2 + z^2$ *and the plane* $x = 4$ *if the density function is* $\rho(x, y, z) = \sqrt{y^2 + z^2}$.

EXAMPLE 3. *Use a triple integral to find the volume of the solid bounded by the surfaces* $z = x^2 + y^2$ *and* $z = 5 - 4x^2 - 4y^2$.

EXAMPLE 4. *Use a triple integral to find the volume of the solid bounded by the elliptic cylinder* $4x^2 + z^2 = 4$ *and the planes* $y = 0$ *and* $y = z + 2$.

Table 1: **Triple integrals over a general bounded region E**

A solid region of **TYPE I**:	A solid region of **TYPE II**:	A solid region of **TYPE III**:
$E = \{(x,y,z) \mid (x,y) \in D, \phi_1(x,y) \le z \le \phi_2(x,y)\}$ where D is the projection of E onto the xy-plane.	$E = \{(x,y,z) \mid (y,z) \in D, \phi_1(y,z) \le x \le \phi_2(y,z)\}$ where D is the projection of E onto the yz-plane.	$E = \{(x,y,z) \mid (x,z) \in D, \phi_1(x,z) \le y \le \phi_2(x,z)\}$ where D is the projection of E onto the xz-plane.
A type 1 solid region 	 A type 2 region	 A type 3 region
$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,\mathrm{d}z\right]\mathrm{d}A$	$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint \left[\int f(x,y,z)\,\mathrm{d}x\right]\mathrm{d}A$	$\iiint_E f(x,y,z)\,\mathrm{d}V =$

When we set up a triple integral it is wise to draw **two** diagrams: one of the solid region E and one of its projection on the corresponding coordinate plane.