14.2: Line Integrals

Line integrals on plane: Let C be a plane curve with parametric equations:

$$x = x(t), y = y(t), \quad a \le t \le b,$$

or we can write the parametrization of the curve as a vector function:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \quad a \le t \le b.$$

DEFINITION 1. *The line integral of $f(x, y)$ with respect to arc length, or the* **line integral of f along C** *is*

$$\int_C f(x, y)\, \mathrm{d}s$$

Recall that the *arc length* of a curve given by parametric equations $x = x(t), y = y(t), \quad a \le t \le b$ can be found as

$$L = \int_a^b \mathrm{d}s,$$

where

$$\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\, \mathrm{d}t.$$

The line integral is then

$$\int_C f(x, y)\, \mathrm{d}s =$$

If we use the vector form of the parametrization we can simplify the notation up noticing that

$$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

and then

$$\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\, \mathrm{d}t =$$

Using this notation the line integral becomes,

$$\int_C f(x, y)\, \mathrm{d}s = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\, \mathrm{d}t.$$

REMARK 2. The value of the line integral does not depend on the parametrization of the curve, provided that *the curve is traversed exactly once as t increases from a to b.*

Let us emphasize that $\quad \mathrm{d}s = |r'(t)|\, \mathrm{d}t = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, \mathrm{d}t.$

EXAMPLE 3. *Evaluate the line integral* $\int_C y\,\mathrm{d}s$, *where* $C : x = t^3, y = t^2, 0 \le t \le 1$.

Line integrals in space: Let C be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \le t \le b,$$

or

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

The **line integral of** f **along** C is

$$\int_C f(x, y, z)\,\mathrm{d}s = \int_a^b f(x(t), y(t), z(t))|r'(t)|\,\mathrm{d}t.$$

Here

$$\mathrm{d}s = |r'(t)|\,\mathrm{d}t = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,\mathrm{d}t.$$

EXAMPLE 4. *Evaluate the line integral* $\int_C (x + y + z)\,\mathrm{d}s$, *where* C *is the line segment joining the points* $A(-1, 1, 2)$ *and* $B(2, 3, 1)$.

Physical interpretation of a line integral: Let $\rho(x,y,z)$ represents the linear density at a point (x,y,z) of a thin wire shaped like a curve C. Then the **mass** m of the wire is:

$$m = \int_C \rho(x,y,z)\,\mathrm{d}s.$$

EXAMPLE 5. *A thin wire with the linear density $\rho(x,y) = x^2 + 2y^2$ takes the shape of the curve C which consists of the arc of the circle $x^2 + y^2 = 1$ from $(1,0)$ to $(0,1)$. Find the mass of the wire.*

Line integrals with respect to $x, y,$ and z. Let C be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \le t \le b,$$

The **line integral of f with respect to** x is,

$$\int_C f(x,y,z)\,\mathrm{d}x = \int_a^b f(x(t),y(t),z(t))x'(t)\,\mathrm{d}t.$$

The **line integral of f with respect to** y is,

$$\int_C f(x,y,z)\,\mathrm{d}y = \int_a^b f(x(t),y(t),z(t))y'(t)\,\mathrm{d}t.$$

The **line integral of f with respect to** z is,

$$\int_C f(x,y,z)\,\mathrm{d}z =$$

These two integral often appear together by the following notation:

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

or

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

EXAMPLE 6. *Compute*

$$I = \int_C -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y,$$

where C is the circle $x^2+y^2=1$ oriented in the counterclockwise direction.

Line integrals of vector fields.

PROBLEM: Given a continuous force field,

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k},$$

such as a gravitational field. Find the work done by the force $\mathbf{F}$ in moving a particle along a curve

$$C: \quad \mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

SOLUTION:

DEFINITION 7. *Let* $\mathbf{F}$ *be a continuous vector field defined on a curve* C *given by a vector function* $\mathbf{r}(t)$*,* $a \le t \le b$*. Then the* **line integral of F along** C *is*

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t) = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t.$$

REMARK 8. Note that this integral depends on the curve orientation:

$$\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t) = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t)$$

EXAMPLE 9. *Find the work done by the force field* $\mathbf{F}(x, y, z) = \langle xy, yz, xz \rangle$ *in moving a particle along the curve* $C : \mathbf{r}(t) = \langle t, t^2, t^3 \rangle, 0 \le t \le 1$.

Relationship between line integrals of vector fields and line integrals with respect to $x, y,$ and z.

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t) =$$