

14.3: The fundamental Theorem for Line Integrals

14.4: Green's Theorem

- Conservative vector field.

DEFINITION 1. A vector field \mathbf{F} is called a **conservative vector field** if it is the gradient of some scalar function f s.t $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for \mathbf{F} .

REMARK 2. Not all vector fields are conservative, but such fields do arise frequently in Physics.

Illustration: Gravitational Field: By Newton's Law of Gravitation the magnitude of the gravitational force between two objects with masses m and M is The gravitational force acting on the object at (x, y, z) is

$$|\mathbf{F}| = G \frac{mM}{r^2},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the objects and G is the gravitational constant.

Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z \rangle$. Then

$$r =$$

Then the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z \rangle$ is

$$\mathbf{F}(x, y, z) =$$

EXAMPLE 3. Let

$$f(x, y, z) = \frac{GmM}{\sqrt{x^2 + y^2 + z^2}}.$$

Find its gradient and answer the questions:

- (a) Is the gravitational field conservative?
- (b) What is a potential function of the gravitational field?

• **The fundamental Theorem for Line Integrals:** Recall Part 2 of the Fundamental Theorem of Calculus:

$$\int_a^b F'(x) dx = F(b) - F(a),$$

where F' is continuous on $[a, b]$.

Let C be a smooth curve given by $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables and ∇f is continuous on C . Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof.

REMARK 4. If C is a closed curve then

COROLLARY 5. *If F is a conservative vector field and C is a curve with initial point A and terminal point B then:*

EXAMPLE 6. *Find the work done by the gravitational field*

$$\mathbf{F}(x, y, z) = -\frac{GmM}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle$$

in moving a particle with mass m from the point $(1, 2, 2)$ to the point $(3, 4, 12)$ along a piecewise-smooth curve C .

Notations And Definitions:

DEFINITION 7. A *piecewise-smooth curve* is called a **path**.

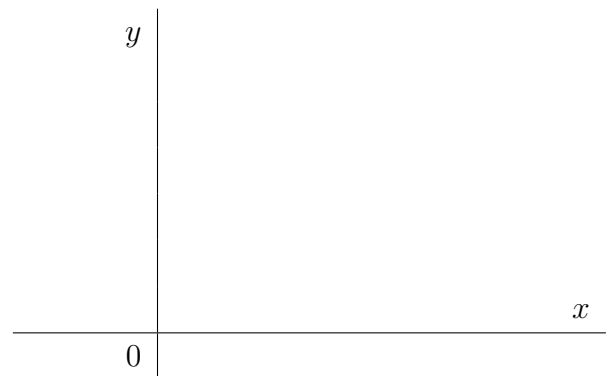
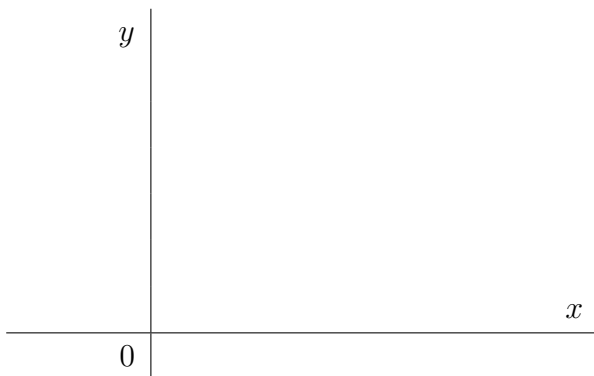
- **Types of curves:**

simple not closed	not simple not closed	simple closed	not simple, closed
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- **Types of regions:**

simply connected	not simply connected
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- **Convention:** The **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C . If C is given by $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \leq t \leq b$, then the region D bounded by C is always on the left as the point $\mathbf{r}(t)$ traverses C .



- The positively oriented boundary curve of D is denoted by ∂D .

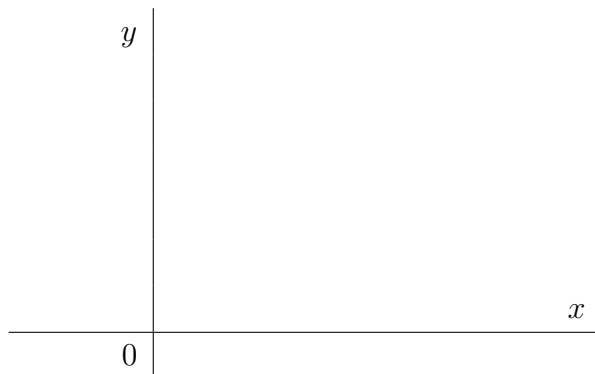
•**GREEN'S THEOREM:** Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C . If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region that contains D , then

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

EXAMPLE 8. Evaluate:

$$I = \oint_C e^x(1 - \cos y) dx - e^x(1 - \sin y) dy$$

where C is the boundary of the domain $D = \{(x, y) : 0 \leq x \leq \pi, 0 \leq y \leq \sin x\}$.



EXAMPLE 9. Let C be a triangular curve consisting of the line segments from $(0,0)$ to $(5,0)$, from $(5,0)$ to $(0,5)$, and from $(0,5)$ to $(0,0)$. Evaluate the following integrals:

$$(a) I_1 = \oint_C \left(x^2y + \frac{1}{2}y^2\right) dx + \left(xy + \frac{1}{3}x^3 + 3x\right) dy$$



$$(b) I_2 = \oint_C \left(x^2y + \frac{1}{2}y^2 + e^{x \sin x}\right) dx + \left(xy + \frac{1}{3}x^3 + x - 4 \arctan(e^y)\right) dy$$

$$(c) I_3 = \oint_C \left(x^2y + \frac{1}{2}y^2 - 55 \arcsin(\sec x)\right) dx + \left(12y^5 \cos y^3 + xy + \frac{1}{3}x^3 + x\right) dy$$

•Application: Computing areas.

$$A(D) = \oint_{\partial D} x dy = - \oint_{\partial D} y dx = \frac{1}{2} \oint_{\partial D} x dy - y dx.$$

EXAMPLE 10. Find the area enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

SUMMARY: Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on an open simply connected domain D . Suppose that P and Q have continuous partial derivatives through D . Then the facts below are equivalent.

$$\boxed{\text{The field } \mathbf{F} \text{ is conservative on } D} \iff \boxed{\text{There exists } f \text{ s.t. } \nabla f = \mathbf{F}}$$

$$\boxed{\text{The field } \mathbf{F} \text{ is conservative on } D} \iff \boxed{\int_{A\tilde{B}} \mathbf{F} \cdot d\mathbf{r} \text{ is independent of path in } D}$$

$$\boxed{\text{The field } \mathbf{F} \text{ is conservative on } D} \iff \boxed{\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \text{ throughout } D}$$

$$\boxed{\text{The field } \mathbf{F} \text{ is conservative on } D} \iff \boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for every closed curve } C \text{ in } D}$$

EXAMPLE 11. Determine whether or not the vector field is conservative:

(a) $\mathbf{F}(x, y) = \langle x^2 + y^2, 2xy \rangle$.

(b) $\mathbf{F}(x, y) = \langle x^2 + 3y^2 + 2, 3x + ye^y \rangle$

EXAMPLE 12. Given $\mathbf{F}(x, y) = \sin y\mathbf{i} + (x \cos y + \sin y)\mathbf{j}$.

(a) Show that \mathbf{F} is conservative.

(b) Find a function f s.t. $\nabla f = \mathbf{F}$

(c) Find the work done by the force field \mathbf{F} in moving a particle from the point $(3, 0)$ to the point $(0, \pi/2)$.

(d) Evaluate $\oint_C \mathbf{F} \, d\mathbf{r}$ where C is an arbitrary path in \mathbb{R}^2 .

EXAMPLE 13. *Given*

$$\mathbf{F} = \langle 2xy^3 + z^2, 3x^2y^2 + 2yz, y^2 + 2xz \rangle.$$

Find a function f s.t. $\nabla f = \mathbf{F}$