14.7: Surface Integrals

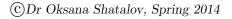
Problem: Find the **mass** of a thin sheet (say, of aluminum foil) which has a shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$.

Solution:

If S is given by $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, $(u,v) \in D$, then the surface integral of f over the surface S is:

$$\iint_S f(x, y, z) \, \mathrm{d}S = \iint_D f(\mathbf{r}(u, v)) |\mathbf{N}(u, v)| \, \mathrm{d}A =$$

EXAMPLE 1. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 \le 2x$, if its density is a function $\rho(x, y, z) = x^2 + y^2 + z^2$.



• Oriented surfaces. We consider only two-sided surfaces.

Let a surface S has a tangent plane at every point (except at any boundary points). There are two unit normal vectors at (x, y, z): $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$.

If it is possible to choose a unit normal vector $\hat{\mathbf{n}}$ at every point (x, y, z) of a surface S so that $\hat{\mathbf{n}}$ varies continuously over S, then S is called **oriented surface** and the given choice of $\hat{\mathbf{n}}$ provides S with an **orientation**. There are two possible orientations for any orientable surface:

• Surface integrals of vector fields.

DEFINITION 2. If \mathbf{F} is a continuous vector field defined on an oriented surface S with unit normal vector $\hat{\mathbf{n}}$, then the surface integral of \mathbf{F} over S is

$$\iint_{S} \mathbf{F} \cdot d\mathbf{S} = \iint_{S} \mathbf{F} \cdot \hat{\mathbf{n}} dS.$$

This integral is also called the flux of F across S.

Note that if S is given by $\mathbf{r}(u, v)$, $(u, v) \in D$, then

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} =$$

and

$$\mathrm{d}\mathbf{S} =$$

Finally,

$$\iint_{S} \mathbf{F} \cdot d\mathbf{S} =$$

EXAMPLE 3. Find the flux of the vector field

$$\mathbf{F} = \left\langle x^2, y^2, z^2 \right\rangle$$

across the surface

$$S = \left\{ z^2 = x^2 + y^2, 0 \le z \le 2 \right\}.$$

EXAMPLE 4. Evaluate $I = \iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle z, y, x \rangle$ and S is the unit sphere centered at the origin.