14.8: STOKES' THEOREM

Stokes' Theorem can be regarded as a 3-dimensional version of Green's Theorem:

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \iint_D \mathrm{curl}\mathbf{F}\cdot \mathbf{k}\, \mathrm{d}A.$$

Let S be an oriented surface with unit normal vector $\hat{\mathbf{n}}$ and with the boundary curve C (which is a space curve).

The orientation on S induces the **positive orientation of the boundary curve** C: if you walk in the positive direction around C with your head pointing in the direction of $\hat{\mathbf{n}}$, then the surface will always be on your left.

The positively oriented boundary curve of an oriented surface S is often written as ∂S.

Stokes' Theorem: *Let S be an oriented piece-wise-smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_S \mathrm{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S},$$

or

$$\iint_S \mathrm{curl}\mathbf{F}\cdot \hat{\mathbf{n}}\, \mathrm{d}S = \oint_{\partial S} \mathbf{F}\cdot \mathrm{d}\mathbf{r}.$$

EXAMPLE 1. *Find the work performed by the forced field* $\mathbf{F}(x, y, z) = \langle 3x^8, 4xy^3, y^2x\rangle$ *on a particle that traverses the curve* C *in the plane* $z = y$ *consisting of 4 line segments from* $(0, 0, 0)$ *to* $(1, 0, 0)$*, from* $(1, 0, 0)$ *to* $(1, 3, 3)$*, from* $(1, 3, 3)$ *to* $(0, 3, 3)$*, and from* $(0, 3, 3)$ *to* $(0, 0, 0)$*.*

EXAMPLE 2. *Verify Stokes' Theorem* $\iint_S \operatorname{curl}\vec{F}\cdot \mathrm{d}\vec{S} = \int_{\partial S}\vec{F}\cdot \mathrm{d}\vec{r}$ *for the vector field* $\vec{F} = \langle 3y, 4z, -6x\rangle$ *and the paraboloid* $z = 9 - x^2 - y^2$ *that lies above the plane* $z = -7$ *and oriented upward. Be sure to check and explain the orientations.*

Solution: Use the following steps:

•Parametrize the boundary circle ∂S and compute the line integral.

•Parametrize the surface of the paraboloid and compute the surface integral:

THEOREM 3. *If* $\mathbf{F}$ *is a vector field defined on* $\mathbb{R}^3$ *whose component functions have continuous partial derivatives and* $\text{curl}\mathbf{F} = \mathbf{0}$, *then* $\mathbf{F}$ *is a conservative vector field.*

SUMMARY: Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ *be a continuous vector field in* $\mathbb{R}^3$.

There exists f s.t. $\nabla f = \mathbf{F}$

$\int_{\breve{AB}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path

$\mathbf{F}$ is conservative in $\mathbb{R}^3$

$\text{curl}\mathbf{F} = \mathbf{0}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C