

## REVIEW

**11.1: Three-dimensional Coordinate System****Cylindrical surfaces**

Note that in  $\mathbb{R}^2$  the graph of the equation involving  $x$  and  $y$  is a curve. In  $\mathbb{R}^3$  an equation in  $x, y, z$  represents a **surface**. (It does not mean that we can't graph curves in  $\mathbb{R}^3$ .)

EXAMPLE 1. *Sketch the graph of  $x^2 + y^2 - 1 = 0$  in  $\mathbb{R}^2, \mathbb{R}^3$ .*

*An equation that contains only two of the variables  $x, y, z$  represents a **cylindrical surface** in  $\mathbb{R}^3$ .  
How to graph cylindrical surface:*

- 1. graph the equation in the coordinate plane of the two variables that appear in the given equation;*
- 2. translate that graph parallel to the axis of the missing variable.*

EXAMPLE 2. *Sketch the graph of  $(x + 2)^2 + (y - 4)^2 = 1$  in  $\mathbb{R}^3$*

EXAMPLE 3. Sketch the graph of  $y = x^2$  in  $\mathbb{R}^3$

EXAMPLE 4. Let  $S$  be the graph of  $x^2 + z^2 - 10z + 21 = 0$  in  $\mathbb{R}^3$ .

(a) Describe  $S$ .

(b) The intersection of  $S$  with the  $xz$ - plane is \_\_\_\_\_

(c) The intersection of  $S$  with the  $yz$ - plane is \_\_\_\_\_

(d) The intersection of  $S$  with the  $xy$ - plane is \_\_\_\_\_

## Spheres

- **Distance formula in  $\mathbb{R}^3$ :** The distance between the points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

EXAMPLE 5. Show that the equation  $x^2 + y^2 + z^2 + 2x - 4y + 8z + 17 = 0$  represents a sphere, and find its center and radius.

In general, completing the squares in

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

produces an equation of the form

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = k$$

- If  $k > 0$  then the graph of this equation is \_\_\_\_\_
- If  $k = 0$ , then the graph is \_\_\_\_\_
- If  $k < 0$  then \_\_\_\_\_

**Regions in  $\mathbb{R}^3$** 

EXAMPLE 6. Describe the set of all points in  $\mathbb{R}^3$  whose coordinates satisfy the following inequality:  
 $x^2 + y^2 < 16$

EXAMPLE 7. Describe the following region:  $\{(x, y, z) | 9 \leq x^2 + y^2 + z^2 \leq 16\}$

**11.2: Vectors and the Dot Product in Three Dimensions**

DEFINITION 8. A **3-dimensional vector** is an ordered triple  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Given the points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$ , the vector  $\mathbf{a}$  with representation  $\overrightarrow{PQ}$  is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

The representation of the vector that starts at the point  $O(0, 0, 0)$  and ends at the point  $P(x_1, y_1, z_1)$  is called the **position** vector of the point  $P$ .

**Vector Arithmetic:** Let  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ .

- Scalar Multiplication:  $\alpha \mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle, \alpha \in \mathbb{R}$ .

- Addition:  $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

Two vectors  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$  are parallel if one is a scalar multiple of the other, i.e. there exists  $\alpha \in \mathbb{R}$  s.t.  $\mathbf{b} = \alpha\mathbf{a}$ . Equivalently:

$$\mathbf{a} \parallel \mathbf{b} \Leftrightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}$$

**The magnitude or length** of  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ :

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

**Zero vector**:  $\mathbf{0} = \langle 0, 0, 0 \rangle$ ,  $|\mathbf{0}| = 0$ .

Note that  $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$ .

**Unit vector in the same direction as  $\mathbf{a}$** :  $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$  *The process of multiplying a vector  $\mathbf{a}$  by the reciprocal of its length to obtain a unit vector with the same direction is called **normalizing  $\mathbf{a}$** .*

Note that in  $\mathbb{R}^2$  a nonzero vector  $\mathbf{a}$  can be determined by its length and the angle from the positive  $x$ -axis:

In  $\mathbb{R}^2$  and  $\mathbb{R}^3$  a vector can be determined by its length and a vector in the same direction:

$$\mathbf{a} = |\mathbf{a}| \hat{\mathbf{a}},$$

i.e.  $\mathbf{a}$  is equal to its length times a unit vector in the same direction.

**EXAMPLE 9.** Find the components of a vector  $\mathbf{a}$  of length  $\sqrt{5}$  that extends along the line through the points  $M(2, 5, 0)$  and  $N(0, 0, 4)$ .

**Standard Basis Vectors**:  $\mathbf{i} = \langle 1, 0, 0 \rangle$ ,  $\mathbf{j} = \langle 0, 1, 0 \rangle$ ,  $\mathbf{k} = \langle 0, 0, 1 \rangle$

Note that  $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$  and

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

**Dot Product** of two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  is a NUMBER:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos \theta,$$

where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ ,  $0 \leq \theta \leq \pi$ .

If  $\mathbf{a} = \mathbf{0}$  or  $\mathbf{b} = \mathbf{0}$  then  $\mathbf{a} \cdot \mathbf{b} = 0$ .

Component Formula for dot product of  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ :

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

If  $\theta$  is the *angle* between two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

DEFINITION 10. Two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  are called **perpendicular** or **orthogonal** if the angle between them is  $\theta = \pi/2$ .

For two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$

$$\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$$

and

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

EXAMPLE 11. Suppose that  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in  $\mathbb{R}^3$ . Find the maximal possible and minimal possible values of dot product  $\mathbf{v} \cdot \mathbf{u}$  among all vectors  $\mathbf{u}$  and  $\mathbf{v}$  such that  $|\mathbf{u}| = 1$  and  $|\mathbf{v}| = 5$ . Make a conclusion.

DEFINITION 12. The **work** done by a force  $\mathbf{F}$  in moving an object from point  $A$  to point  $B$  is given by

$$W = \mathbf{F} \cdot \mathbf{D}$$

where  $\mathbf{D} = \overrightarrow{AB}$  is the distance the object has moved (or displacement).

EXAMPLE 13. A force of  $\mathbf{F} = 7\mathbf{i} - \mathbf{j} + 8\mathbf{k}$  Newtons is applied to a point that moves a distance of 10 meters in the direction of the vector  $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ . How much work is done?

### 11.3: Cross product

Determinant of  $2 \times 2$  and  $3 \times 3$  matrices.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3** is defined by

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 \end{aligned}$$

or copy the first two columns onto the end and then multiply along each diagonal and add those that move from left to right and subtract those that move from right to left.

- THE CROSS PRODUCT IN COMPONENT FORM:

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

REMARK 14. The cross product requires both of the vectors to be three dimensional vectors.

REMARK 15. The result of a dot product is a number and the result of a cross product is a VECTOR!!!

To remember the cross product component formula use the fact that the cross product can be represented as the determinant of order 3:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

**Properties:**

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$(\alpha \mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\alpha \mathbf{b}) = \alpha(\mathbf{a} \times \mathbf{b}), \quad \alpha \in \mathbb{R}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$$

EXAMPLE 16. If  $\mathbf{a} = \langle -2, 1, 1 \rangle$ ,  $\mathbf{b} = \langle 3, 5, 0 \rangle$ , and  $\mathbf{c} = \langle -4, 2, 2 \rangle$  compute each of the following:

(a)  $\mathbf{a} \times \mathbf{b}$

(b)  $\mathbf{b} \times \mathbf{a}$

(c)  $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$

(d)  $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b})$

(e)  $\mathbf{a} \times \mathbf{c}$

- **GEOMETRIC INTERPRETATION OF THE CROSS PRODUCT:**

Let  $\theta$  be the angle between the two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$ ,  $0 \leq \theta \leq \pi$ . Then

1.  $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \sin \theta =$  the area of the parallelogram determined by  $\mathbf{a}$  and  $\mathbf{b}$ ;
2.  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ ;
3. the direction of  $\mathbf{a} \times \mathbf{b}$  is determined by “right hand” rule: if the fingers of your right hand curl through the angle  $\theta$  (which is less than  $\pi$ ) from  $\mathbf{a}$  to  $\mathbf{b}$ , then your thumb points in the direction of  $\mathbf{a} \times \mathbf{b}$ .

**FACT:**  $\mathbf{a} \parallel \mathbf{b} \Leftrightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}$ .



EXAMPLE 17. Given the points  $A(1, 0, 0)$ ,  $B(1, 1, 1)$  and  $C(2, -1, 3)$ . Find

(a) the area of the triangle determined by these points.

(b) Find a unit vector  $\hat{\mathbf{n}}$  orthogonal to the plane that contains the points  $A$ ,  $B$ ,  $C$ .

- **SCALAR TRIPLE PRODUCT** of the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Note that the scalar triple product is a NUMBER.

**FACTS:**

1.  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

2. If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ ,  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$  and  $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$  then  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

3.  $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \text{the volume of the parallelepiped determined by } \mathbf{a}, \mathbf{b}, \mathbf{c}.$

EXAMPLE 18. Determine if the vectors

$$\mathbf{a} = \langle 0, 1, 1 \rangle, \quad \mathbf{b} = \langle 1, 4, -7 \rangle, \quad \mathbf{c} = \langle 2, -1, 4 \rangle$$

are coplanar.<sup>1</sup> If they are not coplanar then find the volume of the parallelepiped that has the given vectors as adjacent edges.

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<sup>1</sup>that is they lie in the same plane