13.2: Iterated integrals

Suppose that $f(x,y)$ is integrable over the rectangle $R=[a,b]\times[b,c]$.

Partial integration of f with respect to x: $\int_a^b f(x,y)\,\mathrm{d}x$

Partial integration of f with respect to y: $\int_c^d f(x,y)\,\mathrm{d}y$

EXAMPLE 1.

$$\int_0^4 (x+3y^2)\,\mathrm{d}x =$$

$$\int_1^4 e^{xy}\,\mathrm{d}y =$$

Iterated integrals:

$$\int_a^b \left[\int_c^d f(x,y)\,\mathrm{d}y\right]\mathrm{d}x = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x$$

and

$$\int_c^d \left[\int_a^b f(x,y)\,\mathrm{d}x\right]\mathrm{d}y = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 2. *Evaluate the integrals:*

$$I_1 = \int_0^1 \int_1^4 x\sqrt{y}\,\mathrm{d}y\mathrm{d}x, \quad I_2 = \int_1^4 \int_0^1 x\sqrt{y}\,\mathrm{d}x\mathrm{d}y$$

FUBINI's THEOREM: *If f is continuous on on the rectangle*

$$R = [a, b] \times [c, d]$$

then

$$\iint_R f(x, y)\,\mathrm{d}A = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 3. *Evaluate*

$$\iint_R x\cos(xy)\,\mathrm{d}A$$

where $R = [-\pi/2, \pi/2] \times [1, 5]$

EXAMPLE 4. (see Section 13.1, Example 3) *Find the volume of the solid S lying under the circular paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2, 2] \times [-3, 3]$.*

FACT: If g and h are continuous functions of one variable and $R = [a, b] \times [c, d]$ then

$$\iint_R g(x)h(y)\,\mathrm{d}A = \left(\int_a^b g(x)\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right).$$

EXAMPLE 5. *If $R = [0, \ln 2] \times [0, \ln 5]$ find* $\iint_R e^{2x-y}\,\mathrm{d}A$.